Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

Silly 10 Point Bonus: (a) State the definition of differentiability for a function of two variables. [You may either state the definition found in the text or the one given by the instructor in class.] (b) Then using only the definition you stated, show the function

$$f(x,y) = xy$$

is differentiable at any point (x,y) in the plane.

(a) To save space, we'll use h instead of Δx and k instead of Δy.

Anton's 8th VERSION: A function, f, of two variables is differentiable at (x,y) if the gradient of f at exists at (x,y) and

(*) $$\lim_{(h,k)\to(0,0)} \frac{f(x+h,y+k) - f(x,y) - \nabla f(x,y)\cdot\langle h,k\rangle}{\|\langle h,k\rangle\|} = 0.$$

[Convince yourself that this is really equivalent to the book's definition.]

Old Anton 4th Edition: A function, f, of two variables is differentiable at (x,y) if the gradient of f at exists at (x,y) and there are two functions

$$\varepsilon_1(h,k) \text{ and } \varepsilon_2(h,k)$$

defined near (0,0) with

$$\varepsilon_1(h,k)\to 0 \text{ and } \varepsilon_2(h,k)\to 0 \text{ as } (h,k)\to(0,0),$$

such that

(**) $$f(x+h,y+k) - f(x,y) = \nabla f(x,y)\cdot\langle h,k\rangle + \langle\varepsilon_1(h,k),\varepsilon_2(h,k)\rangle\cdot\langle h,k\rangle$$

for all (h,k) near (0,0).

(b)

Regardless of which version you use, you should begin with the observation that when $f(x,y) = xy$, both partial derivatives exist at any point (x,y) with

$$\nabla f(x,y) = \langle y, x\rangle.$$

Next, either to deal with the limit in (*) or equation (**) and its mysterious epsilon functions, it is critical to figure out the structure of Δf. This is the easy computation that follows.

(***)
$$\begin{aligned} f(x+h,y+k) - f(x,y) &= (x+h)(y+k) - xy \\ &= yh + xk + hk \\ &= \nabla f(x,y)\cdot\langle h,k\rangle + hk. \end{aligned}$$

Now from (***), the truth of (*) is equivalent to that of

$$\lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = 0.$$

There are a couple of ways to see the truth of this last equation. Obviously, by resorting to a polar squeeze, we have

$$\lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2+k^2}} = \lim_{r\to 0^+} \frac{r\cos(\theta)\, r\sin(\theta)}{r} = \lim_{r\to o^+} \frac{r}{2}\sin(2\theta) = 0.$$

[Yes, there is a squeeze involved.] Alternatively, you might note that

$$0 \le \left| \frac{hk}{\sqrt{h^2+k^2}} \right| \le \frac{\sqrt{h^2+k^2}}{2} \quad since \quad 0 \le (|h| - |k|)^2 = h^2 + k^2 - 2|hk|$$

for a more immediate squeeze.

To complete the argument involving the older, 4th edition definition, it is plain from (**) and (***) on the previous page that we need only find the two mysterious epsilon functions so that

(****) $$\varepsilon_1(h,k)h + \varepsilon_2(h,k)k = hk$$

with

$$\varepsilon_1(h,k) \to 0 \ \ and \ \ \varepsilon_2(h,k) \to 0 \ \ as \ (h,k) \to (0,0),$$

There are two very obvious choices, namely

$$\varepsilon_1(h,k) = k \quad and \quad \varepsilon_2(h,k) = 0$$

or

$$\varepsilon_1(h,k) = 0 \quad and \quad \varepsilon_2(h,k) = h.$$

Either of these obviously performs the appropriate miracle.

In actuality, there is a real continuum of choices: Let $b \in \mathbb{R}$ be fixed. Then we may set

$$\varepsilon_1(h,k) = bk \quad and \quad \varepsilon_2(h,k) = (1-b)h.$$

Onward???