Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

1. (10 pts.)
Let $\mathbf{F}(x,y,z) = \langle x^2, -2, yz\rangle$. Compute the divergence and the curl of the vector field $\mathbf{F}$.

(a)

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(-2) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + y.$$

(b)

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2 & yz \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(-2), -\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(x^2)\right], \frac{\partial}{\partial x}(-2) - \frac{\partial}{\partial y}(x^2) \right\rangle$$

$$= \langle z, 0, 0 \rangle.$$

2. (10 pts.) Convert the given iterated integral into an iterated integral in polar coordinates that has the same numerical value and is easier to evaluate, perhaps. Do not attempt to evaluate the iterated integrals. A picture might help.

$$\int_{-3}^{0}\int_{0}^{(9-x^2)^{1/2}} 2x\,(x^2+y^2)\;dy\,dx = \int_{\pi/2}^{\pi}\int_{0}^{3} 2r\cos(\theta)\,r^2\,r\,dr\,d\theta$$

$$= \int_{\pi/2}^{\pi}\int_{0}^{3} 2r^4\cos(\theta)\;dr\,d\theta.$$

Picture: Quarter disk of radius 3 in the second quadrant bounded by the y-axis on the east and the x-axis on the south.

3. (10 pts.) Write down the triple iterated integral in cylindrical coordinates that would be used to compute the volume of the solid G whose top and bottom are on the sphere defined by the equation $r^2 + z^2 = 100$ and whose lateral boundary is given by the cylinder defined by the equation $r = 5\cdot\sin(\theta)$. Do not attempt to evaluate the integral. (**WARNING:** $r = 5\cdot\sin(\theta)$ has teeth. If you don't pay attention to the details, it bites.)

$$\iiint_G 1\ dV = \iint_R \left[\int_{-(100-r^2)^{1/2}}^{(100-r^2)^{1/2}} 1\ dz \right] dA$$
$$= \int_0^{\pi} \int_0^{5\sin(\theta)} \int_{-(100-r^2)^{1/2}}^{(100-r^2)^{1/2}} 1\ dz\, r\, dr\, d\theta.$$

Note:

$$r = 5\cdot\sin(\theta) \iff r^2 = 5r\cdot\sin(\theta)$$
$$\iff x^2 + y^2 = 5y$$
$$\iff x^2 + \left(y - \tfrac{5}{2}\right)^2 = \left(\tfrac{5}{2}\right)^2.$$

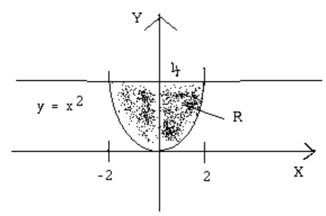
By sketching the rectangular auxiliary $r\theta$-graph, you can see that the circle is traced out as θ runs from 0 to π.

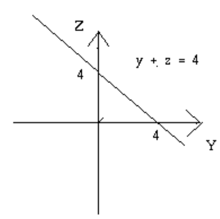
4. (10 pts.) Write down a triple iterated integral in cartesian coordinates that would be used to find the volume of the solid G bounded by the surface $y = x^2$ and the planes $y + z = 4$ and $z = 0$, but do not attempt to evaluate the triple iterated integral you have obtained. [Sketching the traces in the coordinate planes will help.]

$$\iiint_G 1\ dV = \int_{-2}^{2} \int_{x^2}^{4} \int_0^{4-y} 1\ dz\, dy\, dx$$

or

$$= \int_0^4 \int_{-y^{1/2}}^{y^{1/2}} \int_0^{4-y} 1\ dz\, dx\, dy.$$





This, of course, was a silly homework problem.

5. (10 pts.) Write down the triple iterated integral in spherical coordinates that would be used to compute the volume of the solid *G* within the cone defined by $\phi = \pi/4$ and between the spheres defined by $\rho = 4$ and $\rho = 9$ *in the first octant*. Do not attempt to evaluate the interated integrals.

$$\iiint_G 1\, dV = \int_0^{\pi/2}\int_0^{\pi/4}\int_4^9 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

This is an easy to understand spherical wedge. Draw your own side view in the xz-plane. The only wrinkle is the first octant noise that restricts θ so that $0 \le \theta \le \pi/2$.

6. (10 pts.) Compute the surface area of the portion of the hemisphere defined by $z = (16 - x^2 - y^2)^{1/2}$ that lies between the planes defined by $z = 1$ and $z = 2$.

$$\begin{aligned}
\text{SA} &= \iint_R \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)^{1/2} dA \\
&= \iint_R \left(\left(\frac{-x}{(16-x^2-y^2)^{1/2}}\right)^2 + \left(\frac{-y}{(16-x^2-y^2)^{1/2}}\right)^2 + 1\right)^{1/2} dA \\
&= \iint_R \left(\frac{16}{16-x^2-y^2}\right)^{1/2} dA \\
&= \int_0^{2\pi}\int_{\sqrt{12}}^{\sqrt{15}} 4(16-r^2)^{-1/2}\, r\, dr\, d\theta \\
&= \int_{\sqrt{12}}^{\sqrt{15}} 4r(16-r^2)^{-1/2}\, dr \times \int_0^{2\pi} 1\, d\theta \\
&= 4\pi \int_1^4 u^{-1/2}\, du \\
&= 4\pi (2u^{1/2})\Big|_1^4 = 8\pi((4)^{1/2} - (1)^{1/2}) = 8\pi.
\end{aligned}$$

Obviously we have passed to polar coordinates along the way and then used the u-substitution $u = 16 - r^2$ to accomplish our goal. The intersection of the hemisphere with the two planes are a couple of easy to understand circles. These provide the *r* limits of integration.

7. (10 pts.) Compute the work in moving a particle along of the path in the xy-plane that goes from the point (-1,-1) to the point (1,1) along the curve $y = x^5$ against the force field defined by

$$\boldsymbol{F}(x,y) = \left\langle\, y-x \;,\; yx^3 \,\right\rangle.$$

The Toidi's parameterization for the curve, in the correct direction, is given in a vector form by $\mathbf{r}(t) = <t,t^5>$ for t ε [-1,1]. Consequently, $\mathbf{r}'(t) = <1\,,\,5t^4>$ and thus,

$$\begin{aligned}
W &= \int_C \boldsymbol{F}\cdot d\boldsymbol{r} \\
&= \int_C \boldsymbol{F}(\boldsymbol{r}(t))\cdot \boldsymbol{r}'(t)\,dt \\
&= \int_{-1}^{1} (t^5-t)(1) + (t^5t^3)(5t^4)\,dt \\
&= \int_{-1}^{1} t^5 - t + 5t^{12}\,dt \\
&= 10\int_0^1 t^{12}\,dt = \frac{10}{13}.
\end{aligned}$$

How did Em Toidi know that a parameterization is needed?? First, the field is NOT CONSERVATIVE. [Go check this, Frodo.] This means that the Fundamental Theorem of Line Integrals CANNOT BE USED. Second, the curve is NOT CLOSED, although the curve is simple. As a consequence, Green's Theorem CANNOT BE USED. This means finally, you are stuck with the "definition." A nice, easy parameterization is a must. This is actually an easy path integral, oddly. Or was it evenly???? [E.T. used both!!]

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8. (10 pts.) Starting at the point (0,0), a particle goes along the y-axis until it reaches the point (0,4). It then goes from (0,4) to (-4,0) along the circle with equation $x^2 + y^2 = 16$. Finally the particle returns to the origin by travelling along the x-axis. Use Green's Theorem to compute the work done on the particle by the force field defined by $\mathbf{F}(x,y) = <-5y^3,\,5x^3>$ for (x,y) ε $\mathbb{R}^2$. [Draw a picture. This is easy??]

$$\begin{aligned}
W &= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C -5y^3\,dx + 5x^3\,dy \\
&= \iint_R \frac{\partial}{\partial x}(5x^3) - \frac{\partial}{\partial y}(-5y^3)\,dA \\
&= 15\iint_R x^2+y^2\,dA = 15\int_{\pi/2}^{\pi}\int_0^4 r^3\,dr\,d\theta = 480\pi. \quad [\textit{Corrected}.]
\end{aligned}$$

Picture: Quarter disk of radius 4 in the second quadrant bounded by the y-axis on the east and the x-axis on the south. Trace the boundary counter-clockwise.

9. (10 pts.) (a) Show that the vector field

$$\mathbf{F}(x,y) = < \cos(x)e^y , \sin(x)e^y + 10y >$$

is actually a gradient field by producing a function $\phi(x,y)$ such that $\nabla\phi(x,y) = \mathbf{F}(x,y)$ for all (x,y) in the plane.

Evidently

$$\frac{\partial}{\partial y}(\cos(x)e^y) = \cos(x)e^y = \frac{\partial}{\partial x}(\sin(x)e^y + 10y) \textit{ for each } (x,y) \in \mathbb{R}^2.$$

Then

$$\frac{\partial\phi}{\partial x}(x,y) = \cos(x)e^y \Rightarrow \phi(x,y) = \int \cos(x)e^y \, dx$$

$$= \sin(x)e^y + c(y).$$

Therefore,

$$\sin(x)e^y + 10y = \frac{\partial\phi}{\partial y}(x,y) = \frac{\partial}{\partial y}(\sin(x)e^y) + \frac{dc}{dy}(y) \Rightarrow \frac{dc}{dy}(y) = 10y$$

$$\Rightarrow c(y) = 5y^2 + c_0$$

for some number c_0. Hence, $\phi(x,y) = \sin(x)e^y + 5y^2 + c_0$.

(b) Using the Fundamental Theorem of Line Integrals, evaluate the path integral below, where C is any smooth path from the origin to the point $(\pi/2,\ln(2))$. [**WARNING:** You must use the theorem to get any credit here.]

$$\int_C (\cos(x)e^y) \, dx + (\sin(x)e^y + 10y) \, dy = \phi(x,y)\Big|_{(0,0)}^{(\pi/2,\ln(2))}$$

$$= \phi(\pi/2,\ln(2)) - \phi(0,0)$$

$$= \sin(\pi/2)e^{\ln(2)} + 5(\ln(2))^2$$

$$= 2 + 5(\ln(2))^2 = 2 + 5\ln^2(2).$$

10. (10 pts.) Use the substitution $u = (1/2)(x + y)$ and $v = (1/2)(x - y)$ to evaluate the integral below, where R is the bound region enclosed by the triangular region with vertices at (0,0), (2,0), and (1,1).

$$\iint_R \cos\left(\frac{1}{2}(x+y)\right) dA = \iint_R \cos\left(\frac{1}{2}(x+y)\right) dA_{x,y}$$

$$= \int\int_{T^{-1}(R)} \cos(u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{u,v}$$

$$= \int_0^1\int_0^u 2\cos(u)\,dv\,du$$

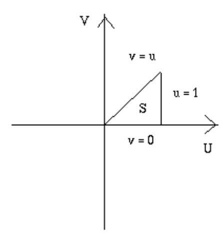
$$= \int_0^1 2u\cos(u)\ du$$

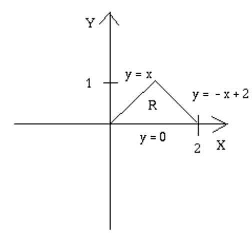
$$= 2u\sin(u)\ \Big|_0^1 - \int_0^1 2\sin(u)\ du$$

$$= 2\sin(1) + 2\cos(1) - 2$$

$\mathbf{T}^{-1}$: $u = (x+y)/2$, $v = (x-y)/2$

$\mathbf{T}$: $x = u+v$,
$y = u-v$





Bounding Curves:
$y = x \quad \Leftrightarrow \quad v = 0$
$y = 2 - x \quad \Leftrightarrow \quad u = 1$
$y = 0 \quad \Leftrightarrow \quad v = u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1)-(1)(1) = -2.$$

Note that $S = T^{-1}(R)$.

Silly 10 Point Bonus: Become a polar explorer. Reveal the details of how one can obtain the exact value of the definite integral

$$\int_0^\infty e^{-x^2}\ dx$$

even though there is no elementary anti-derivative for the function

$$f(x) = e^{-x^2}.$$

Say where your work is, for it won't fit here! [You may gloss some of the technical details related to the matter of convergence.]