Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". **Vector objects must be denoted by using arrows.** Do not "box" your final results. Show me all the magic on the page.

---

1. (10 pts.) Solve the following vector initial-value problem for $\mathbf{y}(t)$:

$$\mathbf{y}''(t) = \mathbf{i} + e^t\mathbf{j}\ ,\ \mathbf{y}(0) = 2\mathbf{i}\ ,\ \textit{and}\ \mathbf{y}'(0) = \mathbf{j}.$$

It follows easily from the vector-valued version of the Fundamental Theorem of Calculus that

$$\mathbf{y}'(t) = \int_0^t \mathbf{y}''(u)\,du + \mathbf{y}'(0)$$
$$= \int_0^t <1, e^u>\,du + <0,1> = <t, e^t-1> + <0,1> = <t, e^t>.$$

Consequently, using the vector-valued version of the Fundamental Theorem again,

$$\mathbf{y}(t) = \int_0^t \mathbf{y}'(u)\,du + \mathbf{y}(0)$$
$$= \int_0^t <u, e^u>\,du + <2,0> = <\frac{t^2}{2}, e^t-1> + <2,0>$$
$$= <\frac{t^2}{2} + 2, e^t-1>.$$

---

2. (10 pts.) Let $\mathbf{r}(t) = <4\cos(t), 4\sin(t), t>$.

Find $\mathbf{T}(\pi/2), \mathbf{N}(\pi/2),\ \textit{and}\ \kappa(\pi/2)$.

Since

$$\mathbf{r}'(t) = <-4\sin(t), 4\cos(t), 1>,\ \textit{and}\ \|\mathbf{r}'(t)\| = \sqrt{17}\ ,$$

it follows that

$$\mathbf{T}(t) = \frac{1}{\sqrt{17}} <-4\sin(t), 4\cos(t), 1>$$

and

$$\mathbf{T}'(t) = \frac{1}{\sqrt{17}} <-4\cos(t), -4\sin(t), 0>\ \textit{and}\ \|\mathbf{T}'(t)\| = \frac{4}{\sqrt{17}}\ .$$

Thus,

$$\mathbf{N}(t) = <-\cos(t), -\sin(t), 0>\ \textit{and}\ \kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{4}{17}\ .$$

Hence,

$$\mathbf{T}(\pi/2) = <-\frac{4}{\sqrt{17}}, 0, \frac{1}{\sqrt{17}}>\ ,\ \mathbf{N}(\pi/2) = <0, -1, 0>\ ,\ \textit{and}\ \kappa(\pi/2) = \frac{4}{17}.$$

3. (10 pts.) Obtain an arc-length parameterization for the curve **r**(t) = < t, 3cos(2t), 3sin(2t) > in terms of the initial point (0, 3, 0) which is the terminal point of **r**(0) in standard position. Rather than overloading the symbol **r**, write this new parameterization as **R**(s). How are **R** and **r** related?

First, since **r**(0) = <0, 3, 0>, the (signed) distance along the curve from the point given by **r**(0) on the graph to that given by **r**(t) is

$$\begin{aligned} s = \varphi(t) &= \int_0^t \|\mathbf{r}'(u)\|\,du \\ &= \int_0^t \|<1,-6\sin(2u),6\cos(2u)>\|\,du \ . \\ &= \int_0^t 37^{1/2}\,du = 37^{1/2}(t-0) = 37^{1/2}t. \end{aligned}$$

Solving for t in terms of s yields

$$t = 37^{-1/2}s \quad \text{so that } \varphi^{-1}(t) = 37^{-1/2}t.$$

Thus,

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}(\varphi^{-1}(s)) = \mathbf{r}(37^{-1/2}s) \\ &= < \frac{s}{37^{1/2}}\ ,\ 3\cos\left(\frac{2s}{37^{1/2}}\right)\ ,\ 3\sin\left(\frac{2s}{37^{1/2}}\right) >. \end{aligned}$$

Evidently, $\mathbf{R}(s) = \mathbf{r}(\varphi^{-1}(s))$, or equivalently, $\mathbf{r}(t) = \mathbf{R}(\varphi(t))$.

4. (10 pts.) A particle moves smoothly in such a way that at a particular time t = 0, we have **v**(0) = < 1 , 2 > and **a**(0) = < 3 , 0 >. If we write **a**(0) in terms of **T**(0) and **N**(0), then

$$\mathbf{a}(0) = a_{\mathbf{T}}(0)\mathbf{T}(0) + a_{\mathbf{N}}(0)\mathbf{N}(0),$$

where

(a) **T**(0) = $\frac{1}{\|\mathbf{v}(0)\|}\mathbf{v}(0) = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$,

(b) $a_{\mathbf{T}}(0)$ = $\mathbf{T}(0) \cdot \mathbf{a}(0) = \frac{3}{\sqrt{5}}$,

(c) $a_{\mathbf{N}}(0)$ = $\sqrt{\|\mathbf{a}(0)\|^2 - (a_{\mathbf{T}})^2} = \frac{6}{\sqrt{5}}$, and

(d) **N**(0) = $\left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$.

(e) Also, $\kappa(0) = \frac{\|<1,2,0>\times<3,0,0>\|}{\|\mathbf{v}(0)\|^3} = \frac{6}{5^{3/2}}$

5. (10 pts.) (a) Find the limit.

$$\lim_{t\to 1}\left\langle \frac{3}{t^2}, \frac{\ln(t)}{t^2-1}, \sin(\frac{\pi}{2}t)\right\rangle = \left\langle 3, \frac{1}{2}, 1\right\rangle$$

The only silliness is the middle limit. Either recognize a loggy derivative limit as part of the mess or use l'Hopital's Rule there.

(b) Find parametric equations for the line tangent to the graph of $\mathbf{r}(t) = (2 - \ln(t))\mathbf{i} + t^2\mathbf{j}$ at the point where $t_0 = 1$.

Since $\mathbf{r}'(t) = < -t^{-1} , 2t >$, $\mathbf{r}(1) = < 2 - \ln(1) , 1 > = < 2 , 1 >$, and $\mathbf{r}'(1) = < -1 , 2 >$. Consequently, an appropriate set of parametric equations is given by x = 2 - t and y = 1 + 2t.

6. (10 pts.) Let

$$f(x,y) = y - x^2 .$$

(a) Obtain an equation for the level curve for f that passes through the point (-1,2).

Since $f(-1,2) = 2 - (-1)^2 = 1$, an equation for the level curve through (-1,2) is $1 = y - x^2$, an equation a garden variety parabola.

(b) Compute the value of the directional derivative

$$D_{\mathbf{u}}f(-1,2)$$

when **u** is the unit vector in the plane that is in the direction of the gradient of f at (-1,2).

When **u** is the unit vector in the plane that is in the direction of the gradient of f at (-1,2), we have

$$D_{\mathbf{u}}f(-1,2) = \|\nabla f(-1,2)\| = \sqrt{5}$$

since

$$\nabla f(x,y) = <-2x,1> \text{ when } f(x,y) = y - x^2 .$$

Note: To compute the directional derivative in most other directions, you might want to have the unit vector in hand. Here, we don't need it thanks to an earlier analysis.

7. (10 pts.) (a) Use an appropriate form of chain rule to find $\partial z/\partial v$ when $z = \sin(x)\sin(y)$ when $x = u + v$ and $y = u^2 - v^2$.

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \cos(x)\sin(y)(1) + \sin(x)\cos(y)(-2v)$$

$$= \cos(u+v)\sin(u^2-v^2) - \sin(u+v)\cos(u^2-v^2)(2v).$$

(b) Assume that $F(x,y,z) = 0$ defines z implicitly as a function of x and y. Show that if $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}.$$

Using classical curly d notation, since x and y are independent variables and z is a function of x and y, we have

$$\frac{\partial F(x,y,z)}{\partial x} = 0 \Rightarrow \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}.$$

8. (10 pts.). (a) Use limit laws and continuity properties to evaluate the following limit.

$$\lim_{(x,y)\to(-1/4,\pi)} (xy^2\sec^2(xy)) = (-\frac{1}{4})(\pi)^2\sec^2(-\frac{\pi}{4}) = -\frac{\pi^2}{2}$$

(b) Evaluate the limit, if it exists, by converting to polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{r\cos(\theta)(r\sin(\theta))^2}{r} = \lim_{r\to 0^+} r^2\cos(\theta)\sin^2(\theta) = 0$$

since

$$-r^2 \leq r^2\cos(\theta)\sin^2(\theta) \leq r^2$$

for a polar squeeze.

9. (10 pts.) (a) Calculate $\partial z/\partial x$ using implicit differentiation when $3x^2 + 4y^2 + \tan(z) = 12$. Leave your answer in terms of x, y, and z.

By pretending z is a function of the two independent variables x and y and performing partial differentiations on both sides of the equation above, we have

$$\frac{\partial}{\partial x}\left[3x^2+4y^2+\tan(z)\right] = \frac{\partial[12]}{\partial x} \Rightarrow 6x + \sec^2(z)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{-6x}{\sec^2(z)}$$

provided z is not an odd integer multiple of $\pi/2$.

(b) Find all second-order partial derivatives for the function $f(x,y) = x^3y^4$. Label correctly.

Since $f_x(x,y) = 3x^2y^4$ and $f_y(x,y) = 4x^3y^3$, we have $f_{xx}(x,y) = 6xy^4$, $f_{yy}(x,y) = 12x^3y^2$, and $f_{xy}(x,y) = f_{yx}(x,y) = 12x^2y^3$.

10. (10 pts.) (a) Compute the total differential dz when

$$z = \tan^{-1}(xy).$$

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{y}{1+(xy)^2}dx + \frac{x}{1+(xy)^2}dy.$$

(b) Assume $f(1,-2) = 4$ and $f(x,y)$ is differentiable at $(1,-2)$ with $f_x(1,-2) = 2$ and $f_y(1,-2) = -3$. Obtain an equation for the plane tangent to the graph of f at $P(1,-2,4)$.

$$z = f(1,-2) + f_x(1,-2)(x-1) + f_y(1,-2)(y-(-2))$$

$$= 4 + 2(x-1) - 3(y-2)$$

Silly 10 Point Bonus: (a) State the definition of differentiability for a function of two variables. [You may either state the definition found in the text or the one given by the instructor in class.] (b) Then using only the definition you stated, show the function

$$f(x,y) = x^2 + y^2$$

is differentiable at any point (x,y) in the plane. Say where your work is, for it won't fit here.

See c3-t2-bo.pdf