Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Generic vector objects must be denoted by using arrows. Since the answer really consists of all the magic transformations, do not "box" your final results. Show me all the magic on the page neatly.

1. (10 pts.) Evaluate the following iterated integral.

$$\int_0^{\ln(2)}\int_0^1 xye^{y^2x}\,dy\,dx = \int_0^{\ln(2)}\int_0^x e^u\,\frac{1}{2}\,du\,dx \quad using\ u = y^2x\ so\ du = 2xy\,dy$$

$$= \frac{1}{2}\int_0^{\ln(2)} e^x - 1\,dx$$

$$= \frac{1}{2}e^{\ln(2)} - \frac{1}{2}e^0 - \frac{1}{2}\ln(2) = \frac{1}{2} - \ln(\sqrt{2}).$$

Attempting to integrate this with the order of integration reversed is a serious mistake.

2. (10 pts.) Convert the given iterated integral into an iterated integral in polar coordinates that has the same numerical value and is easier to evaluate, perhaps. Do not attempt to evaluate the iterated integrals. Assume $a > 0$.

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dy\,dx = \int_0^{\pi/2}\int_0^a \frac{1}{\sqrt{1+r^2}}\,r\,dr\,d\theta$$

Picture: Quarter disk of radius a in the first quadrant bounded by the y-axis on the west and the x-axis on the south.

Silly 10 Point Bonus: Become a polar explorer. Compute the exact value of the double integral

$$\iint_R x\,dA$$

where R is the region in the first quadrant bounded by the lines $y = x/3$ and $y = 2x$ and the circle centered at the origin with a radius of 2.

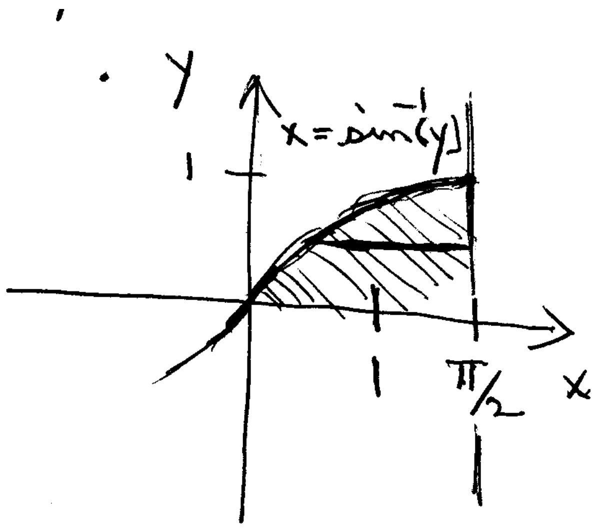
Let α denote the angle between the line defined by $y = x/3$ and the positive x-axis and let β denote the angle between the line defined by $y = 2x$ and the positive x-axis. Then

$$\iint_R x\,dA = \int_\alpha^\beta\int_0^2 r\cos(\theta)\,r\,dr\,d\theta = \frac{8}{3}(\sin(\beta) - \sin(\alpha)) = \frac{8}{3}\left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}}\right)$$

since $\tan(\beta) = 2$ and $\tan(\alpha) = 1/3$. Lay out a couple of right triangles and let Pythagoras clean up the mess. [I've skipped a couple of *obvious* steps.]

3. (10 pts.) Express the integral as an equivalent integral with the order of integration reversed. Sketching the region is a key piece of the puzzle.

$$\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} f(x,y)\ dxdy \ = \ \int_0^{\pi/2} \int_0^{\sin(x)} f(x,y)\ dydx$$



Observe that we get the information to do the sketch from the limits of integration of the original iterated integral, namely that the region, R, consists of the pairs (x,y) which satisfy $\sin^{-1}(y) \le x \le \pi/2$ when $0 \le y \le 1$. The bounding curves may be read from these inequalities easily. Of course in reversing the integrals, a key piece of the puzzle is that $x = \sin^{-1}(y)$ if, and only if $y = \sin(x)$ for the critical ordered pairs along the top boundary. Thus, $0 \le y \le \sin(x)$ when we have $0 \le x \le \pi/2$.

4. (10 pts.) Set up, but do not attempt to evaluate, a triple iterated integral in cartesian coordinates that would be used to find the volume of the solid G bounded below by the elliptic paraboloid $z = 4x^2 + y^2$ and above by the cylindrical surface $z = 4 - 3y^2$, [Hint: The upper and lower surfaces are on a platter with a cherry on top. Find the projection onto the xy-plane of the curve obtained when the two surfaces intersect.]

$$\iiint_G 1\ dV \ = \ \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} 1\ dzdydx$$

or

$$\iiint_G 1\ dV \ = \ \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{4x^2+y^2}^{4-3y^2} 1\ dzdxdy.$$

will do. When the two surfaces intersect, we have $4x^2 + y^2 = 4 - 3y^2$. Consequently, the projection on the xy-plane of the curve of intersection is given by the set of ordered pairs satisfying the equation $x^2 + y^2 = 1$, an equation for the unit circle centered at the origin of the xy-plane.

5. (10 pts.) Write down the triple iterated integral in cylindrical coordinates that would be used to compute the volume of the solid G whose top is the plane $z = 9$ and whose bottom is the paraboloid $z = x^2 + y^2$. Do not attempt to evaluate the integral.

$$\iiint_G 1 \, dV = \int_0^{2\pi}\int_0^3\int_{r^2}^9 1 \; r\,dz\,dr\,d\theta$$

The projection of the line of intersection of the plane $z = 9$ and the paraboloid $z = x^2 + y^2$ is the circle with radius 3 centered at the origin. I imagine you might be able to picture this in your bio-computer.

6. (10 pts.) Locate all relative extrema and saddle points of the following function.

$$f(x,y) = x^2 + 8y^2 - x^2y$$

Use the second partials test in making your classification. (Fill in the table below after you locate all the critical points.)

Crit.Pt.	f_{xx} @ c.p.	f_{yy} @ c.p.	f_{xy} @ c.p.	D @ c.p.	Conclusion
(x,y)	$2 - 2y$	16	$-2x$	$32-32y-4x^2$	
(0,0)	2	16	0	32	Rel. Min.
(4,1)	0	16	-8	-64	Saddle Pt.
(-4,1)	0	16	8	-64	Saddle Pt.

// Since $f_x(x,y) = 2x(1 - y)$ and $f_y(x,y) = 16y - x^2$, the critical points of f are given by the solutions to the following system:

$$2x(1 - y) = 0 \text{ and } 16y - x^2 = 0.$$

The system is plainly equivalent to

$$x = 0 \text{ and } 16y = x^2,$$

or

$$y = 1 \text{ and } 16y = x^2 .$$

Solving these systems yields three critical points: (0,0), (4,1), and (-4,1). Now go up and fill in the silly table.

7. (5 pts.) Write down the triple iterated integral in spherical coordinates that would be used to compute the volume of the solid G bounded above by the sphere defined by $\rho = 4$ and below by the cone defined by $\phi = \pi/3$. Do not attempt to evaluate the iterated integrals.

$$\iiint_G 1 \, dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

Ouch! It feels like an easy spherical wedgie.

8. (15 pts.) Let $f(x,y) = x^2y^2$ on the closed unit disk defined by $x^2 + y^2 \leq 1$. Find the absolute extrema and where they occur. Use Lagrange multipliers to analyze the function on the boundary. Do not neglect the interior of the disk in doing your analysis!!

// First, we deal with interior points of the disk. Since $\nabla f(x,y) = < 2xy^2 , 2yx^2 >$, there are infinitely many critical points in the interior of the disk where either component of the pair (x,y) is zero. Obviously, $f(x,y) = 0$ if either $x = 0$ or $y = 0$.

Now to study f on the boundary using Lagrange multipliers, set $g(x,y) = x^2 + y^2 - 1$. Then (x,y) is on the circle defined by $x^2 + y^2 = 1$ precisely when (x,y) satisfies $g(x,y) = 0$. Since $\nabla g(x,y) = <2x, 2y>$, $\nabla g(x,y) \neq <0, 0>$ when (x,y) is on the circle given by $g(x,y) = 0$. Plainly f and g are smooth enough to satisfy the hypotheses of the Lagrange Multiplier Theorem. Thus, if a constrained local extremum occurs at (x,y), there is a number λ so that $\nabla f(x,y) = \lambda\nabla g(x,y)$. Now

$$\nabla f(x,y) = \lambda\nabla g(x,y) \Rightarrow < 2xy^2 , 2yx^2 > = \lambda< 2x,2y >$$

$$\Rightarrow 0 = 2xy(y - x)(y + x)$$

$$\Rightarrow x = 0 \text{ or } y = 0 \text{ or } y = x \text{ or } y = -x$$

by performing a little routine algebraic magic. Solving each of the systems consisting of

(a) $x^2 + y^2 = 1$ and $x = 0$, (b) $x^2 + y^2 = 1$ and $y = 0$,

(c) $x^2 + y^2 = 1$ and $y = x$, (d) $x^2 + y^2 = 1$ and $y = -x$,

yields the eight (8) desired (constrained) critical points. Since the Extreme-value Theorem ensures we have absolute extrema, we only need to know the function values at these points. Here are the solutions to the systems:

$$(a): \ (0, \pm 1)$$

$$(b): \ (\pm 1, 0)$$

$$(c): \ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$(d): \ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

If (x,y) is an (a) or (b) pair, then $f(x,y) = 0$, the minimum value. Taking into account the interior analysis, we see the minimum actually occurs on the x- and y- axis points in the closed disk. If (x,y) is a (c) or (d) pair, then $f(x,y) = 1/4$, the maximum value.//

9. (10 pts.) Reveal all the details in using the substitution $u = x/a$ and $v = y/b$ with $a > 0$ and $b > 0$ to evaluate the integral below, where R is the elliptical region defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1.$$

$$\iint_R 1\, dA = \iint_R 1\, dA_{x,y} = \int\!\!\int_{T^{-1}(R)} 1 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{u,v} = \iint_S ab\, dA_{u,v}$$

$$= (ab)Area(\,S\,) = ab\pi,$$

since $S = T^{-1}(R)$ is the closed disk of radius 1 centered at the origin in the uv-plane.

T^{-1} : $u = x/a$, $v = y/b$ T : $x = au$, $y = bv$

Bounding Curve: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff u^2 + v^2 = 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = (a)(b)-(0)(0) = ab$$

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10. (10 pts.) Compute the surface area of the portion of the paraboloid defined by $z = 16 - x^2 - y^2$ that lies above the xy-plane.

$$\begin{aligned}
\text{SA} &= \iint_R \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right)^{1/2} dA \\
&= \iint_R ((-2x)^2 + (-2y)^2 + 1)^{1/2}\, dA \\
&= \iint_R (1 + 4x^2 + 4y^2)^{1/2}\, dA \\
&= \int_0^{2\pi}\int_0^4 (1+4r^2)^{1/2}\, r\,dr\,d\theta \\
&= \int_0^4 r(1+4r^2)^{1/2}\, dr \times \int_0^{2\pi} 1\, d\theta \\
&= 2\pi \int_1^{65} u^{1/2}\,\frac{1}{8}\,du \\
&= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{65} = \frac{\pi}{6}\left((65)^{3/2} - (1)^{3/2}\right).
\end{aligned}$$

Obviously we have passed to polar coordinates along the way and then used the u-substitution $u = 1 + 4r^2$ to accomplish our goal. The intersection of the paraboloid with the xy-plane is an easy to understand circle centered at the origin with radius 4. This provides the r and θ limits of integration.

Silly 10 Point Bonus: Become a polar explorer. Compute the exact value of the double integral

$$\iint_R x\, dA$$

where R is the region in the first quadrant bounded by the lines $y = x/3$ and $y = 2x$ and the circle centered at the origin with a radius of 2. [A brief indication of an answer is on the bottom of Page 1 of 5.]