Student Number: Exam Number:

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be very careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all your magic on the page.*

1. (120 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(15 pts./part)

(a) $\dfrac{dr}{d\theta} + r \cdot \tan(\theta) = \cos(\theta)$; $r(0) = 10$

(b) $y' = \cos(x) \cdot (y^2 - 1)$

1. (c) $11y' + x^{-1} \cdot y = 12xy^{-10}$; $y(1) = 4$

1. (d) $y'' + y = \csc(x)$

1. (e) $[\sec(x)\tan(x) - 8y^2]dx + [20e^{10y} - 16xy]dy = 0$

1. (f) $x^2 \cdot y'' - x \cdot y' = 4 \cdot \ln(x)$

1\. (g) $(x^2 + xy + 1y^2)dx - (x^2)dy = 0$

1\. (h) $\dfrac{d^4y}{dx^4} - 16y = 0$

2. (25 pts.) (a) Obtain the recurrence formula for the power series solution at $x_0 = 0$ of the homogeneous O.D.E. $y'' - xy = 0$.

(b) Compute the first five (5) coefficients of the unique solution $y_1(x)$ that satisfies the initial conditions $y_1(0) = 1$ and $y_1'(0) = 1$.

(c) Compute the first five (5) coefficients of the unique solution $y_2(x)$ that satisfies the initial conditions $y_2(0) = 0$ and $y_2'(0) = 1$.

(d) Using the Wronskian, show that the set $\{y_1 , y_2\}$ is linearly independent. [Hint: You really do have enough data here to deal with this!!! Look carefully at (b) and (c).]

(e) Your computation in (d) shows that the set $\{y_1 , y_2\}$ is a fundamental set of solutions. This means that the unique solution $f(x)$ to the ODE $y'' - xy = 0$ that satisfies the initial conditions $f(0) = a$ and $f'(0) = b$ can be written as a linear combination of the functions y_1 and y_2 thus:

$$f(x) = c_1 \cdot y_1(x) + c_2 \cdot y_2(x)$$

for all x near zero. What are the values of the coefficients c_1 and c_2 in terms of a and b???

3. (15 pts.) It is known that $f(x) = x^r$ is a solution of the homogeneous linear differential equation

(*) $$x^2y'' - 5xy' + 9y = 0$$

for a particular value of r.

(a) Find the value of r by substituting f(x) into (*), obtaining an algebraic equation in r, and solving the equation involving r.

(b) Then find a second, linearly independent solution to (*) by using the technique of reduction of order.

4. (10 pts.) Work the following problem which uses Hooke's law:

Be sure to state what your variables represent using complete sentences.

// A 128 pound stone is attached to the lower end of a spring with a fixed support. (The spring is vertical.) The weight stretches the spring 1 foot when in its equilibrium position. If the weight is then pushed up 4 inches and released at time t = 0 with an initial velocity of 8 inches per second directed downward, obtain the displacement as a function of time. [Assume free, undamped motion.] //

5. (10 pts.) The equation $x^2 \cdot y'' + x(x - 1)y' - 3y = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved. [Hint: Use ALL the information you have available after solving the indicial equation. Write those power series varmints right carefully folks.]

6. (10 pts.) A large water tank initially contains 100 gallons of brine in which 25 pounds of salt is dissolved. Starting at time t = 0 minutes, a brine solution containing 5 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute. The mixture is kept uniform by a mixer which stirs it continuously, and the well-stirred mixture flows out at the same rate. When will the tank have a mixture containing 100 pounds of salt???? Explain. Details are essential here!!

8. (10 pts.) Use only the Laplace transform machine to completely solve the following initial value problem.

$$y' = f(t) \text{ , where } f(t) = \begin{cases} \sin(t) & \text{, for } 0 \le t < \pi \\ 0 & \text{, for } \pi \le t \end{cases}$$

and $y(0) = 0$.

Silly 20 point bonus: Show how to use the Laplace transform to evaluate the following antiderivative:

$$\int 2t \cdot e^t \cdot \sin(t)dt$$

[Tell me where your work is, for you do not have room here. Do not waste time trying "parts" on this varmint. Even if you integrate this successfully using "parts" you will only get credit if you use the Laplace machine. The point of the problem is that the Laplace transform provides an appropriate route to deal with this ... if you know how to hold your mouth sinistrally.]