1. (a) $y' + 2xy = 8x$; $y(0) = 10$

This is linear as written. If you carefully follow the recipe and deal with the initial condition, you will get $y(x) = 4 + 6\cdot\exp(-x^2)$.

(b) $2\cdot(y^2 - 1)dx + 20\cdot\sec(x)dy = 0$

This is separable as written. Look. Because (b) is equivalent to $2\cdot(y^2 - 1) + 20\cdot\sec(x)(dy/dx) = 0$, you get a couple of constant solutions from the zeros of $y^2 - 1$: $y(x) = 1$ and $y(x) = -1$. By separating variables and cleaning up the algebra, you get $\int 2\cos(x)dx + \int 20\cdot(y^2 - 1)^{-1}dy = c$. The first integral is cheap thrills and the second may be handled by doing a partial fraction decomposition. A one-parameter family of solutions is given by $2\cdot\sin(x) + 10\cdot\ln|y - 1| - 10\cdot\ln|y + 1| = c$. [Explicit solutions are accessible, but take more time than you have.]

(c) $(y^2 + xy + x^2)dx - (xy)dy = 0$

This is a homogeneous equation, with each coefficient function homogeneous of degree 2. Letting $y = v\cdot x$, you must deal with the following:

$$\int [v/(v + 1)]dv - \int x^{-1}dx = c.$$

Integrating will produce $v - \ln|v + 1| - \ln|x| = c$. You may then finish this by replacing v above with y/x.

(d) $7y' + x^{-1}y = 12x^2y^{-6}$ for $x > 0$

This is clearly a Bernoulli equation. Turn it into a linear equation using the substitution $v = y^7$. Yadda, yadda, yadda. $y^7 = 3x^3 + cx^{-1}$ etc.

(e) $(x + y - 8)dx + (x - y + 4)dy = 0$

This is one of the equations with linear coefficient functions that may be reduced either to a simple homogeneous equation or a slightly messy separable varmint. Since the lines defined by the equations $x + y - 8 = 0$ and $x - y + 4 = 0$ intersect at the point $(h,k) = (2,6)$, the substitution $x = X + 2$ and $y = Y + 6$ results in the homogeneous equation $(X + Y)dX + (X - Y)dY = 0$. Thankfully, this is also exact. Here's a one-parameter family of solutions:

$$X^2 + 2XY - Y^2 = c.$$

After replacing X and Y, we have

$$(x - 2)^2 + 2(x - 2)\cdot(y - 6) - (y - 6)^2 = c.$$

Why you should check for exactness first!!

Here is a second, more immediate solution: Equation (e) is exact. Doing the trivial integrations in your head, which is not advised, leads to the following one-parameter family of solutions after clearing the common denominator:

$$x^2 + 2xy - y^2 - 16x + 8y = C$$

(f) $y' = f(x)$, where $f(x) = \begin{cases} 2x & , \text{ for } 0 \le x < 3 \\ 6 & , \text{ for } 3 \le x \end{cases}$

and $y(0) = 1$.

Linear ... with an integrating factor $\mu = 1$ ... ugh; so gluing the pieces together, we get

$$y(x) = \begin{cases} x^2 + 1 & , \text{ for } 0 \le x < 3 \\ 6x - 8 & , \text{ for } 3 \le x \end{cases}$$

2. (10 pts.) (a) Obtain the differential equation and initial condition needed to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. [For (c), the exact value in terms of natural logs will suffice.]

//A tank initially contains 50 gallons of pure water. Starting at time t = 0, a brine containing 2 pounds of dissolved salt per gallon flows into the tank at a rate of 3 gallons per minute. Suppose the mixture is kept uniformly mixed by constant stirring and flows out of the tank at the same rate at which it enters. When will the tank contain 10 pounds of dissolved salt?//

(a) Let $x(t)$ denote the number of pounds of NaCl in the tank at time t, in minutes. Then $x'(t) = 6 - 3(x(t)/50)$, and $x(0) = 0$.

(b) The differential equation may be viewed as separable or linear. Since we get an explicit solution more readily by treating the varmint as linear, we shall do so. [Later, in Chapter 4, you will learn how to handle this animal without doing any integrations!!]

Here are the details. The DE, $x'(t) + 3(x(t)/50) = 6$, is linear with an integrating factor consisting of $\mu(t) = e^{(3/50)t}$. Thus, performing the usual incantations and deftly passing our writing utensils over the cellulose, we obtain the solution to the I.V.P.:

$$x(t) = 100 - 100e^{-(3/50)t}.$$

(c) Solving $10 = x(t_0)$ yields $t_0 = (50/3)\cdot(\ln(10/9)$.

Lagniappe: To get a crude idea of how large this is without using a calculator, it suffices to observe that

$$\ln(10/9) = \int_9^{10} (1/t)dt$$

is easy to estimate since $1/10 \le 1/t \le 1/9$ when t is in the interval [9,10]. Consequently $5/3 < t_0 < 50/27$ in minutes. This means that you get 10 pounds in under two minutes! [Your calculator will produce $t_0 \approx 1.75....$]