
General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

---

1. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$(m - \pi)^3(m - (2+i))^2(m - (2-i))^2 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**] (b) (5 pt.) What is the order of the differential equation?

(a):

$$y = c_1e^{\pi x} + c_2xe^{\pi x} + c_3x^2e^{\pi x} + c_4e^{2x}\sin(x) + c_5e^{2x}\cos(x)$$
$$+ c_6xe^{2x}\sin(x) + c_7xe^{2x}\cos(x)$$

(b): The equation is a seventh order ODE.

---

2. (15 pts.) Given that $f(x) = \sin(2x)$ is a solution of the homogeneous linear O.D.E. $y'' + 4y = 0$, using only the method of reduction of order, obtain a second, linearly independent solution. [**WARNING: No reduction, no credit!! Show all steps of this neatly while using notation correctly.**]

Substitution of $y = v\cdot\sin(2x)$ into the equation and doing a little algebra yields $0 = \sin(2x)v'' + 4\cdot\cos(2x)v'$. Letting $w = v'$, and doing a bit more algebra allows us to obtain $w' + 4\cdot\cot(2x)w = 0$, a linear homogeneous first order equation with integrating factor $\mu = \sin^2(2x)$. By using this appropriately, we get $w = c\cdot\csc^2(2x)$. Thus $v = -(c/2)\cdot\cot(2x) + d$. Consequently, by setting $c = -2$ and $d = 0$, we obtain $y = \cos(2x)$ ... no surprise, this.

---

3. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular solution, y_p, for the O.D.E.

$$y'' - y' = 10x^2 - 7\sin(x) - 32xe^x.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

First, the corresponding homogeneous equation is

$$y'' - y' = 0,$$

which has an auxiliarly equation given by $0 = m^2 - m = m(m-1)$. Thus a fundamental set of solutions for the corresponding homogeneous equation is $\{ 1, e^x \}$. Taking this into account, we may now write

$$y_p = Ax^3 + Bx^2 + Cx + D\cdot\sin(x) + E\cdot\cos(x) + Fxe^x + Gx^2e^x,$$

or something equivalent.

---

4. (15 pts.) **Using the method of variation of parameters, not the method of undetermined coefficients**, find a particular integral, y_p, of the differential equation

$$y'' - y = 10e^x.$$

[**Hint: Read this problem twice and do exactly what is asked to avoid heartbreak!!** *Do not obtain* y_p *using the method of undetermined coefficients. Do not waste time getting the general solution*.]

Corresponding Homogeneous: $y'' - y = 0$ F.S. = $\{e^x, e^{-x}\}$.
If $y_p = v_1e^x + v_2e^{-x}$ then v_1' and v_2' are solutions to the following system:

$$ex\cdot v_1' + e^{-x}v_2' = 0$$

$$e^x\cdot v_1' - e^{-x}v_2' = 10e^x.$$

Solving the system yields $v_1' = 5$ and $v_2' = -5e^{2x}$. Thus, by integrating, we obtain $v_1 = 5x + c$ and $v_2 = -(5/2)e^{2x} + d$. Consequently, $y_p = v_1\cdot e^x + v_2e^{-x} = 5xe^x - (5/2)e^x$, after cleaning up things. [You may 'drop' $-(5/2)e^x$ from y_p. Why??]

5. (15 pts.) Write down the general solution to each of the following linear constant coefficient homogeneous equations.

(a) $y'' - 10y' + 25y = 0$ Solution: $y = c_1e^{5x} + c_2xe^{5x}$

(b) $y'' - y' - 20y = 0$ Solution: $y = c_1e^{5x} + c_2e^{-4x}$

(c) $d^4y/dx^4 + 9(dy^2/dx^2) = 0$

Solution: $y = c_1 + c_2x + c_3\sin(3x) + c_4\cos(3x)$

---

6. (10 pts.) Very carefully obtain the general solution to the following Euler-Cauchy O.D.E.:

$$x^2y''(x) - 4xy'(x) + 6y(x) = 8\cdot \ln(x)$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) - 5w'(t) + 6w(t) = 8\cdot t$$

The corresponding homogeneous equ.: $w''(t) - 5w'(t) + 6w(t) = 0$
The auxiliary equation: $(m - 2)\cdot(m - 3) = 0$
Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ e^{2t}, e^{3t} \}$
The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that $w_p(t) = (4/3)t + (10/9)$ is a particular integral. Consequently, the general solution to the original ODE, the one involving y, is

$$y(x) = c_1x^2 + c_2x^3 + (4/3)\ln(x) + (10/9).$$

Silly 10 Point Bonus: To see that the set of functions consisting of $\{ x^3 , |x|^3 \}$ is linearly independent, you must use the definition of linear independence. Pretend that we have $c_1x^3 + c_2|x|^3 = 0$ for every real number x. Then, replacing x with -1 gives us $-c_1 + c_2 = 0$, and substituting 1 for x yields $c_1 + c_2 = 0$. The truth of these two equations implies that $c_1 = c_2 = 0$. Thus, the two functions $f(x) = x^3$ and $g(x) = |x|^3$ are linearly independent. A routine computation that treats separately the cases where $x > 0$, $x < 0$, and finally $x = 0$ will reveal that the Wronskian of f and g is identically zero. [Do the computations!] Now if the set { f, g } with $f(x) = x^3$ and $g(x) = |x|^3$ were a fundamental set of solutions to a linear homogeneous equation of the form $a_0(x)y'' + a_1(x)y' + a_1(x)y = 0$, where a_0, a_0, and a_0 are continuous real functions defined on all of $\mathbb{R}$ with $a_0 \neq 0$ on all of $\mathbb{R}$, then from Theorem 4.4 of Section 4.1, or Theorem 4.17 of Section 4.6, the Wronskian of f and g would have to be nonzero somewhere on $\mathbb{R}$. [This answers the question easily for 'reasonable' linear ODE's. What about the case where we allow coefficient functions to be discontinuous??]

7. (15 pts.) Suppose

$$y(x) = \sum_{k=0}^{\infty} c_k x^k$$

is a solution of the homogeneous second order linear equation $y'' - 10x^2y = 0$. (a) Obtain the recurrence formula for the coefficients of $y(x)$. (b) Which coefficients must be zero?? (c) If $y(x)$ also satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$, what are the values of c_2, c_3, c_4, and c_5??

After you substitute $y(x)$ above into the D.E. and clean up the algebra by doing a little re-indexing, you obtain the following:

$$0 = 2c_2 \quad + \quad 6c_3x \quad + \quad \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - 10c_{k-2}]x^k$$

From this you can deduce that $c_2 = 0$, $c_3 = 0$, and that for $k \geq 1$, we have $c_{k+2} = 10c_{k-2}/[(k+2)(k+1)]$. [Note: As a consequence, it is easy to prove by induction that $c_{4j+2} = c_{4j+3} = 0$ for j any non-negative integer. You don't have to get this fancy.] For part (c), it is easy to see that $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5 = 1/2$.

---

8. (10 pts.) Obtain the solution to the following initial value problem:

$$y'' - y' = 2\cdot\sin(x)$$

$$y(0) = -1 \text{ , and } y'(0) = 1$$

The corresponding homogeneous equation: $y'' - y' = 0$
The auxiliary equation: $0 = m^2 - m = m(m-1)$.
A fundamental set of solutions to the homogeneous: $\{ 1, e^x \}$.
Set $y_p = A\cdot\sin(x) + B\cdot\cos(x)$. By substituting y_p into the original ODE and using the linear independence of your friendly sine and cosine demons, you can magically produce a system of linear linear equations that A and B must satisfy:

$$-A - B = 0$$

$$-A + B = 2$$

Solving this yields $A = -1$ and $B = 1$. It follows that the general solution to the ODE resembles something like

$$y(x) = c_1 + c_2e^x - \sin(x) + \cos(x).$$

Finally, using the initial conditions here will lead to yet another linear system in c_1 and c_2, which when solved, will provide you the solution to the IVP:

$$y(x) = -4 + 2e^x - \sin(x) + \cos(x).$$

Silly 10 Point Bonus: Show that the following set of two functions is linearly independent and yet cannot be a fundamental set of solutions for any homogeneous second order linear O.D.E. on the whole real line: $\{ x^3 , |x|^3 \}$. Say where your work is! [Look on the bottom of page 3.]