
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. In particular, be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.*

---

1. (10 pts.) (a) Suppose that f(t) is defined for t > 0. What is the definition of the Laplace transform of f, $\mathfrak{L}\{f(t)\}$, in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 & , \quad 5 \geq t > 0 \\ 3 & , \quad t > 5 \ . \end{cases}$$

$\mathfrak{L}\{f(t)\}(s) = (3/s)e^{-5s}$, $s > 0$, with much weeping and gnashing of teeth. The intermediate magical steps are critical. You could check your "final answer" by using number 15 on the table of Laplace Transforms!!

---

2. (15 pts.) Very neatly transform the given initial value problem into a linear system in $\mathfrak{L}\{x\}$ and $\mathfrak{L}\{y\}$ and stop. Do not attempt to solve for $\mathfrak{L}\{x\}$ or $\mathfrak{L}\{y\}$. Do not take inverse transforms, and do not attempt to combine terms over a common denominator.

$$\text{I.V.P.:} \quad x'(t) + y(t) = 8\cdot\sin^2(5t)$$

$$y'(t) - x(t) = 5\cdot\delta(t - 40)$$

$$x(0) = 1 \ , \ y(0) = -1$$

After performing the transformation two-step, you might waltz right up to

$$s\mathfrak{L}\{x\} + \mathfrak{L}\{y\} = [4/s] - [(4s)/(s^2 + 100)] + 1$$

$$-\mathfrak{L}\{x\} + s\mathfrak{L}\{y\} = 5e^{-40s} - 1$$

after applying the identity $\sin^2(x) = [1 - \cos(2x)]/2$ in your sleep.

3. (15 pts.) (a) If f(t) and g(t) are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, (f*g)(t)??

$$(f*g)(t) = \int_0^t f(x)g(t - x)\, dx$$

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute (f*g)(t) when f(t) = 24t and g(t) = $3t^2$.

$$(f*g)(t) = \int_0^t f(x)g(t - x)\, dx = \int_0^t 24x\cdot(3)(t - x)^2\, dx$$

$$= \int_0^t 72xt^2 - 144x^2t + 72x^3\, dx$$

$$= 36t^4 - 48t^4 + 18^4 = 6t^4$$

(c) Compute the Laplace transform of f*g when $f(t) = 4t\cdot\cos(t)$ and $g(t) = te^{-4t}$. [Do not attempt to simplify the algebra after computing the transform.]

$$\mathfrak{L}\{(f*g)(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\cdot\mathfrak{L}\{g(t)\}(s)$$

$$= 4\mathfrak{L}\{t\cdot\cos(t)\}(s)\cdot\mathfrak{L}\{te^{-4t}\}(s)$$

$$= 4(s^2 - 1)\cdot(s^2 + 1)^{-2}\cdot(s + 4)^{-2}$$ or equivalent.

---

4. (10 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathfrak{L}\{y(t)\}(s) = \frac{se^{-9s}}{s^2 + 81} - \frac{8\cdot s}{(s - 4)^2 + 36}.$$

What's the solution, y(t) , to the IVP??

$$y(t) = u_9(t)\cos(9(t-9)) - 8e^{4t}\cos(6t) - (32/6)e^{4t}\sin(6t)$$

after just a little of the usual prestidigitation --- taking the inverse transform of both sides, using linearity, then factoring unity correctly, and invoking the avatar of zero who transmogrifies the toad to princely table form.

Bonkers Bonus : Obviously f and g are not unique. Why? Remember that convolution is commutative. Here are a couple of possible choices: f(t) = sin(t) and g(t) = h(t), or f(t) = h(t) and g(t) = sin(t). You need only solve the silly thing and remember that $\mathfrak{L}^{-1}\{\mathfrak{L}\{h(t)\}(s)\}(t) = h(t)$. Additionally using the linearity of all participants and being willing to tinker, you can find infinitely many of the varmints.

5. (10 pts.) The equation $x^2 \cdot y'' + x(x + 3)y' - 8y = 0$ has a regular singular point at $x_0 = 0$. (a) Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. (b) Then use all the information available to say what the two linearly independent solutions provided by Theorem 6.3 look like without attempting to obtain the coefficients of the power series involved.// The indicial equation is $r(r-1) + 3r - 8 = (r + 4)(r - 2) = 0$. You can, of course, obtain this in a couple of ways.

$$y_1(x) = |x|^2 \sum_{k=0}^{\infty} c_k x^k$$

$$y_2(x) = |x|^{-4} \sum_{k=0}^{\infty} d_k x^k + C y_1(x) \cdot \ln|x|$$

6. (15 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y'(t) = f(t) \text{ , where } f(t) = \begin{cases} 2t & \text{, for } 0 \le t < 3 \\ 6 & \text{, for } 3 \le t \end{cases}$$

and $y(0) = 1$.

Observe that $f(t) = 2t - 2(t - 3)u_3(t)$. Thus, applying our friendly Laplace transform to both sides of the differential equation, using the initial condition, and solving for the Laplace transform of y yields

$$\mathfrak{L}\{y(t)\}(s) = [2/(s^3)] - [2e^{-3s}/(s^3)] + [1/s].$$

Thus, after not bowing at all to the partial fraction proprietor, you may write

$$y(t) = t^2 - u_3(t)(t-3)^2 + 1$$

Finally, after you march up and down the unit steps a few times, you have

$$y(t) = \begin{cases} t^2 + 1 & \text{, for } 0 \le t < 3 \\ 6t - 8 & \text{, for } 3 \le t \end{cases}$$

more or less. There are, of course, a couple of inequalities that we have fudged. [Look at Test 1, Problem 1(f)...!!??]

7. (10 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^4 - 2x^3 + x^2)y'' + 2(x - 1)y' + x^2 y = 0$$

Since, $P_1(x) = 2(x-1)/[x^2(x-1)^2]$ and $P_2(x) = x^2/[x^2(x-1)^2]$, doing the necessary thinking or attempting to compute suitable limits reveals that 0 is an irregular singular point of the DE, and 1 is a regular singular point.

8. (15 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $$h(t) = \begin{cases} 5 & , 0 < t < 3 \\ -9 & , 3 < t < 8 \\ 2 & , 8 < t \,. \end{cases}$$

$$= 5\cdot u_0(t) + (-9 - (5))\cdot u_3(t) + (2 - (-9))\cdot u_8(t)$$
$$= 5\cdot u_0(t) - 14\cdot u_3(t) + 11\cdot u_8(t)$$

$$\mathfrak{L}\{h(t)\}(s) = 5\mathfrak{L}\{u_0(t)\}(s) - 14\mathfrak{L}\{u_3(t)\}(s) + 11\mathfrak{L}\{u_8(t)\}(s)$$

$$= (1/s)(5 - 14e^{-3s} + 11e^{-8s})$$

(b) $g(t) = 2\cdot t\cdot e^{3t}\cdot \sin(t)$

$$\mathfrak{L}\{g(t)\}(s) = 2\mathfrak{L}\{e^{3t}(t\cdot \sin(t))\}(s)$$

$$= 2\mathfrak{L}\{t\cdot \sin(t)\}(s - 3)$$

$$= 2(2)\cdot(s - 3)/[\,(\,(s - 3)^2 + 1^2\,)^2\,]$$

$$= 4(s - 3)/[((s - 3)^2 + 1)^2]$$

This can also be handled by following the line of transformation that begins $\mathfrak{L}\{g(t)\}(s) = 2\mathfrak{L}\{t(e^{3t}\cdot \sin(t))\}(s)$ and uses differentiation ... 'tis much messier though.

(c) $$f(t) = \begin{cases} 10 & , 0 < t < 2 \\ 5t & , 2 < t \end{cases}$$

$$= 10 + (5t - 10)\cdot u_2(t)$$

$$\mathfrak{L}\{f(t)\}(s) = \mathfrak{L}\{10\}(s) + \mathfrak{L}\{(5t - 10)\cdot u_2(t)\}(s)$$

$$= 10s^{-1} + \mathfrak{L}\{g(t-2)\cdot u_2(t)\}(s), \text{ where } g(t-2) = 5t - 10$$

$$= 10s^{-1} + e^{-2s}\mathfrak{L}\{g(t)\}(s), \text{ with } g(t) = 5(t+2) - 10$$

$$= 10s^{-1} + e^{-2s}\mathfrak{L}\{5t\}(s)$$

$$= 10s^{-1} + 5e^{-2s}s^{-2}$$

Silly 10 Point Bonus: Pretend the function h(t) is continuous and bounded for $t \geq 0$. The solution to the initial value problem, $y''(t) + y(t) = h(t)$ with $y(0) = a$ and $y'(0) = b$, is

$$y(t) = (f*g)(t) + a\cdot \cos(t) + b\cdot \sin(t)$$

for a couple of functions f(t) and g(t). Identify f and g. [**Warning**: Say where your work is below, for you do not have room to solve this in the space remaining on this page. Details are a must. There will be no credit otherwise.]