Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. In particular, be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.*

1. (10 pts.) (a) Suppose that f(t) is defined for $t > 0$. What is the definition of the Laplace transform of f, $\mathcal{L}\{f(t)\}$, in terms of a definite integral??

$\mathcal{L}\{f(t)\}(s) =$

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 , & 5 \geq t > 0 \\ 3 , & t > 5 . \end{cases}$$

$\mathcal{L}\{f(t)\}(s) =$

2. (15 pts.) Very neatly transform the given initial value problem into a linear system in $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ and stop. Do not attempt to solve for $\mathcal{L}\{x\}$ or $\mathcal{L}\{y\}$. Do not take inverse transforms, and do not attempt to combine terms over a common denominator.

I.V.P.:

$$x'(t) + y(t) = 8 \cdot \sin^2(5t)$$

$$y'(t) - x(t) = 5 \cdot \delta(t - 40)$$

$$x(0) = 1 , \ y(0) = -1$$

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3. (15 pts.) (a) If f(t) and g(t) are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, (f*g)(t)??

(f*g)(t) =

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute (f*g)(t) when $f(t) = 24t$ and $g(t) = 3t^2$.

(f*g)(t) =

(c) Compute the Laplace transform of f*g when $f(t) = 4t \cdot \cos(t)$ and $g(t) = te^{-4t}$. [Do not attempt to simplify the algebra after computing the transform.]

$\mathfrak{L}\{(f*g)(t)\}(s) =$

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4. (10 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathfrak{L}\{y(t)\}(s) = \frac{se^{-9s}}{s^2 + 81} - \frac{8 \cdot s}{(s - 4)^2 + 36} .$$

What's the solution, y(t) , to the IVP??

y(t) =

5. (10 pts.) The equation $x^2 \cdot y'' + x(x + 3)y' - 8y = 0$ has a regular singular point at $x_0 = 0$. (a) Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. (b) Then use all the information available to say what the two linearly independent solutions provided by Theorem 6.3 look like without attempting to obtain the coefficients of the power series involved.

$y_1(x) =$

$y_2(x) =$

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6. (15 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y'(t) = f(t) \text{ , where } f(t) = \begin{cases} 2t & \text{, for } 0 \le t < 3 \\ 6 & \text{, for } 3 \le t \end{cases}$$

and $y(0) = 1$.

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7. (10 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^4 - 2x^3 + x^2)y'' + 2(x - 1)y' + x^2y = 0$$

8. (15 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $h(t) = \begin{cases} 5 & , 0 < t < 3 \\ -9 & , 3 < t < 8 \\ 2 & , 8 < t . \end{cases}$

$\mathfrak{L}\{h(t)\}(s) =$

(b) $g(t) = 2 \cdot t \cdot e^{3t} \cdot \sin(t)$

$\mathfrak{L}\{g(t)\}(s) =$

(c) $f(t) = \begin{cases} 10 & , 0 < t < 2 \\ 5t & , 2 < t \end{cases}$

$\mathfrak{L}\{f(t)\}(s) =$

Silly 10 Point Bonus: Pretend the function h(t) is continuous and bounded for $t \geq 0$. This means that h has a Laplace transform defined for $s > 0$. The solution to the initial value problem, $y''(t) + y(t) = h(t)$ with $y(0) = a$ and $y'(0) = b$, is

$$y(t) = (f*g)(t) + a \cdot \cos(t) + b \cdot \sin(t)$$

for a couple of functions f(t) and g(t). Identify f and g. [**Warning**: Say where your work is below, for you do not have room to solve this in the space remaining on this page. Details are a must. There will be no credit otherwise.]