NAME:

Student Number: Exam Number:

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Since the answer really consists of all the magic transformations and incantations, do not "box" your final results. Show me all the magic on the page.*

1. (120 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(a) $y' = 3x^2(1 - y^2)^{1/2}$; $y(0) = 1/2$

(b) $(2x + 3y)dx + (2x - y)dy = 0$

1. (c) $y'' + y = \cot(x)$

1. (d) $6y' + x^{-1}y = 12xy^{-5}$; $y(1) = 1$

1. (e) $x^2y'' - xy' = 4x^2$ for $x > 0$.

1. (f) $[20e^{10x} - 8y^2]dx + [\sec(y)\tan(y) - 16xy]dy = 0$

2. (10 pts.) Work the following problem which uses Hooke's law:
Be sure to state what your variables represent using complete sentences. Assume that the resistance of the medium may be neglected and that there are no external forces present. [Free, Undamped Motion]
// An 8 pound weight is placed at the end of a coil spring suspended from the ceiling of a lab. After coming to rest in its equilibrium position the weight is set into vertical motion and it is determined that the period of the resulting motion is 4 seconds. After a while, the motion of the weight is stopped, and the 8 pound weight is replaced by another weight. After this weight has come to rest, like the first weight, it is set into vertical motion. It is determined that this second weight has a period of 10 seconds. How much does this second weight weigh?? //

3. (15 pts.)
The equation $x^2y'' + x(x - 5)y' + 9y = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved. [Hint: Use ALL the information you have available after solving the indicial equation. Write those power series varmints right carefully folks.]

4.(15 pts.)
It is known that $f(x) = x^r$ is a solution of the homogeneous linear differential equation

(*) $x^2y'' - 7xy' + 16y = 0$

for a particular value of r.

(a) Find the value of r by substituting f(x) into (*), obtaining an algebraic equation in r, and solving the equation involving r.

(b) Then find a second, linearly independent solution to (*) by using only the technique of reduction of order.

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5. (10 pts.) A large water tank initially contains 100 gallons of brine in which 50 pounds of salt is dissolved. Starting at time t = 0 minutes, a brine solution containing 5 pounds of salt per gallon flows into the tank at the rate of 4 gallons per minute. The mixture is kept uniform by a mixer which stirs it continuously, and the well-stirred mixture flows out at the same rate. When will the tank have a mixture containing 100 pounds of salt???? Explain. Details are essential here!!

6. (15 pts.) Use only the Laplace transform machine to completely solve the following initial value problem.

$$y' = f(t) \text{ , where } f(t) = \begin{cases} -1 & \text{, for } 0 \le t < \pi \\ \cos(t) & \text{, for } \pi \le t \end{cases}$$

and $y(0) = 1$.

7. (15 pts.) (a) Obtain the recurrence formula for the power series solution at $x_0 = 0$ of the homogeneous linear O.D.E. $y'' - x^2 y = 0$.

(b) Compute the first five (5) coefficients of the unique solution $y_1(x)$ that satisfies the initial conditions $y_1(0) = 1$ and $y_1'(0) = 1$. [Work on the back of Page 5 of 6.]

Silly 20 point bonus: (a) It's easy to obtain a linear 3rd order constant coefficient homogeneous ODE with e^x and $\sin(x)$ as solutions. Do so. (b) It's slightly more difficult to obtain a linear 2nd order homogeneous ODE with e^x and $\sin(x)$ as solutions. Do this. [Oh, by the way, the equation is NOT a constant coefficient equation.] Say where your work is.