1. (a) $y' + 2x^{-1}y = 8x$; $y(1) = 10$

This is linear as written. An integrating factor is $\mu = x^2$ for $x > 0$. If you carefully follow the recipe and deal with the initial condition, you will get $y(x) = 2x^2 + 8x^{-2}$ for $x > 0$.

(b) $2\cdot\csc(y)dx + 20\cdot(x^2 - 1)dy = 0$

This is separable as written. Look. Because (b) is equivalent to $2\cdot\csc(y) + 20\cdot(x^2 - 1)y' = 0$, there are no constant solutions since $\csc(y)$ has no zeros. By separating variables and cleaning up the algebra, you get $\int 2(x^2 - 1)^{-1}\,dx + \int 20\cdot\sin(y)dy = c$. The second integral is cheap thrills and the first may be handled by doing a partial fraction decomposition. A one-parameter family of solutions is given by $\ln|x - 1| - \ln|x + 1| - 20\cdot\cos(y) = c$. [Explicit solutions are accessible, but take more time than you have.]

(c) $(2x^2 + y^2)dx + (x^2 - xy)dy = 0$

This is a homogeneous equation, with each coefficient function homogeneous of degree 2. Letting $y = v\cdot x$, you must deal with the following:

$$\int[(v - 1)/(v + 2)]dv - \int x^{-1}dx = c.$$

Integrating will produce $v - 3\cdot\ln|v + 2| - \ln|x| = c$. You may then finish this by replacing v above with y/x.

(d) $5\cdot y' + x^{-1}y = 8\cdot x^2\cdot y^{-4}$ for $x > 0$

This is clearly a Bernoulli equation. Turn it into a linear equation using the substitution $v = y^5$. Yadda, yadda, yadda.
$y^5 = 2x^3 + cx^{-1}$, or more explicitly, $y = (2x^3 + cx^{-1})^{1/5}$.

(e) $(y\cdot\cos(xy) - 6x)dx + (x\cdot\cos(xy) + 8y)dy = 0$

This varmint is plainly exact. A one-parameter family of solutions is given by $\sin(xy) - 3x^2 + 4y^2 = C$, where C is an arbitrary constant.

(f) $y' = f(x)$, where $f(x) = \begin{cases} 6 & , \text{for } 0 \le x < 3 \\ 2x & , \text{for } 3 \le x \end{cases}$

and $y(0) = -3$.

Linear ... with an integrating factor $\mu = 1$ ... ugh; so gluing the pieces together, we get

$$y(x) = \begin{cases} 6x - 3 & , \text{for } 0 \le x < 3 \\ x^2 + 6 & , \text{for } 3 \le x \end{cases}$$

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2. (5 pts.) Very neatly provide the verification that $2x^3 + 6xy^2 = 2$ is an implicit solution of the differential equation $2xy \cdot y' + x^2 + y^2 = 0$ on the interval $0 < x < 1$.

By differentiating $2x^3 + 6xy^2 = 2$ implicitly, pretending y is a function of x, we obtain $6x^2 + 6y^2 + 12xy \cdot y' = 0$. Multiplying both sides of this last equation by 1/6 yields $x^2 + y^2 + 2xy \cdot y' = 0$.

If $0 < x < 1$, then by doing routine algebra, we can readily see that the equation $2x^3 + 6xy^2 = 2$ is equivalent to

$$y^2 = (1/3) \cdot (1 - x^3)/x,$$

and the right side is positive. Consequently, we may solve for y.

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3. (5 pts.) It is known that every solution to the differential equation $y'' - 4 \cdot y = 0$ is of the form

$$y = c_1 \cdot e^{2x} + c_2 \cdot e^{-2x}.$$

Which of these functions satisfies the initial conditions $y(0) = 2$ and $y'(0) = -4$?? [Hint: Determine c_1 and c_2 by solving an appropriate linear system. Don't waste time verifying y, above, is a solution.]

The two initial conditions are equivalent to the linear system $2 = c_1 + c_2$ and $-4 = 2c_1 - 2c_2$. Solving this system yields $c_1 = 0$ and $c_2 = 2$. Consequently, the solution to the IVP is $y(x) = 2e^{-2x}$.

Silly 10 Point Bonus: (a) The Fundamental Theorem of Calculus provides a neat formal solution involving a definite integral with respect to the variable 't' to the following dinky IVP:

$$y'(x) = \exp(x^2) \text{ and } y(0) = 1.$$

What is that solution? (b) Unfortunately $g(x) = \exp(x^2)$ cannot be integrated in elementary terms. Use the answer to (a), the Maclaurin series for e^x, and term-wise integration, to obtain a power series solution to the IVP. [Say where your work is! You don't have room here!]

(a)

$$y(x) = 1 + \int_0^x \exp(t^2)dt \text{ for all } x.$$

(b)

$$\begin{aligned} y(x) &= 1 + \int_0^x \exp(t^2)dt \\ &= 1 + \int_0^x \sum_{k=0}^{\infty} [\frac{(t^2)^k}{k!}]dt \\ &= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{(t^2)^k}{k!}dt \\ &= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{t^{2k}}{k!}dt \\ &= 1 + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!} \text{ for all } x. \end{aligned}$$