General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Be very careful. Remember that what is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate.*

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1. (40 pts.) Solve each of the following second order differential equations or initial value problems. Be very careful. Show all essential work. Do not write nonsense.

(a) $y'' - 4y' - 21y = 0$

Solution: $y = c_1e^{7x} + c_2e^{-3x}$

(b) $d^3y/dx^3 + 9(dy/dx) = 0$

Solution: $y = c_1 + c_2\sin(3x) + c_3\cos(3x)$

(c) $y'' - 18y' + 81y = 0$

Solution: $y = c_1e^{9x} + c_2xe^{9x}$

(d) $y'' - y' = 4e^x$; $y(0) = -2$, $y'(0) = 1$

Solution: $y = 1 - 3e^x + 4xe^x$

Silly 10 Point Bonus: Magically obtain the following antiderivative without integrating by parts or any other way!!

$$\int \sin(x)\cdot e^x dx = y$$

The silly antiderivative equation is equivalent to

$$y' = \sin(x)e^x,$$

a linear, constant coefficient linear ODE with a U.C. driving function. The corresponding homogeneous linear ODE is $y' = 0$, which has as a fundamental set of solutions the singleton, { 1 }, which contains the lowly "one function". Given the fundamental set above and the U.C. driving function, we'd expect to have a particular integral of the form $y_p = A\cdot\sin(x)e^x + B\cdot\cos(x)e^x$. If y_p is to be a particular integral, then, for each x we must have $(A - B)\cdot\sin(x)e^x + (A + B)\cdot\cos(x)e^x = y_p' = \sin(x)e^x$. From the linear independence of $\sin(x)e^x$ and $\cos(x)e^x$, this is equivalent to A and B satisfying the following linear system: $A - B = 1$ and $A + B = 0$. This is equivalent to $A = 1/2$ and $B = -1/2$. Bottom line:

$$\int \sin(x)\cdot e^x dx = c\cdot 1 + (1/2)\sin(x)e^x - (1/2)\cos(x)e^x$$

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2. (5 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m^3(m - \pi)(m - (1+5i))^3(m - (1-5i))^3 = 0$$

(a) (3 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**] (b) (2 pt.) What is the order of the differential equation?

(a):

$$y = c_1 + c_2x + c_3x^2 + c_4e^{\pi x} + c_5e^x\cos(5x) + c_6e^x\sin(5x)$$
$$+ c_7xe^x\cos(5x) + c_8xe^x\sin(5x)$$
$$+ c_9x^2e^x\cos(5x) + c_{10}x^2e^x\sin(5x)$$

(b): The equation is a tenth order ODE. Simply look at the degree of the polynomial making up the left side of the auxiliary equation.

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3. (10 pts.) Given that $f(x) = \sin(2x)$ is a solution of the homogeneous linear O.D.E. $y'' + 4y = 0$, use only the method of reduction of order to find a second, linearly independent solution. [**WARNING: No reduction, no credit!! Show all steps of this neatly while using notation correctly.**]

The substitution of $y = v\cdot\sin(2x)$ into the equation and doing a little algebra yields $0 = \sin(2x)v'' + 4\cdot\cos(2x)v'$. Letting $w = v'$, and doing a bit more algebra allows us to obtain $w' + 4\cdot\cot(2x)w = 0$, a linear homogeneous first order equation with integrating factor $\mu = \sin^2(2x)$. By using this appropriately, we get $w = c\cdot\csc^2(2x)$. Thus , by integrating, we obtain $v = -(c/2)\cdot\cot(2x) + d$. Consequently, by setting $c = -2$ and $d = 0$, we obtain $y = \cos(2x)$ ... no surprise, this.

[Of course you could also choose $c = -2$ and $d = 1$ since the function $y = \sin(2x) + \cos(2x)$ is a linearly independent solution. In fact, as long as c is chosen to be nonzero, you will get a linearly independent solution. Just compute the Wronskian.]

4. (20 pts.) **Using the method of variation of parameters, not the method of undetermined coefficients**, find a particular integral, y_p, of the differential equation

$$y'' - 4y' = 10.$$

[Hint: Read this problem twice and do exactly what is asked to avoid heartbreak!! *Do not obtain* y_p *using the method of undetermined coefficients*.]

Corresponding Homogeneous: $y'' - 4y' = 0$

F.S. = $\{1, e^{4x}\}$.

If $y_p = v_1 \cdot 1 + v_2 e^{4x}$, then v_1' and v_2' are solutions to the following system:

$$1 \cdot v_1' + e^{4x} v_2' = 0$$

$$4e^{4x} v_2' = 10.$$

Solving the system yields $v_1' = -5/2$ and $v_2' = (5/2)e^{-4x}$. Thus, by integrating, we obtain $v_1 = -(5/2)x + c$ and $v_2 = -(5/8)e^{-4x} + d$. Consequently, $y_p = v_1 \cdot 1 + v_2 e^{4x}$ $= -(5/2)x - (5/8)$, after cleaning up things. [You may 'drop' $-(5/8)$ from y_p. Why??]

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5. (5 pts.) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution and perform the reduction, but **do not attempt to solve the separable or homogeneous equation you obtain**.

$$(x - y - 8)dx + (2x - 2y + 2)dy = 0$$

The lines defined by $x - y - 8 = 0$ and $2x - 2y + 2 = 0$ are parallel. Set $z = x - y$. Then $dy = dx - dz$. After substituting and cleaning up the algebra just a little, we have

$$(3z - 6)dx - (2z + 2)dz = 0,$$

which plainly is separable.

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6. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular solution, y_p, for the O.D.E.

$$y'' - 3y' = 10x^2 - 7\sin(x) - 32xe^{3x}.$$

Do not attempt to actually find the numerical values of the coefficients!!

Since a fundamental set of solutions to the corresponding homogeneous equation consists of $\{ 1, e^{3x} \}$,

$y_p = Ax^3 + Bx^2 + Cx + D\cdot\sin(x) + E\cdot\cos(x) + Fxe^{3x} + Gx^2e^{3x}$.

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7. (10 pts.) (a) Obtain the differential equation and initial condition needed to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. This will probably involve a second equation relating dependent and independent variables. [For (c), the exact value in terms of natural logs will suffice.]

//Assume Newton's Law of Cooling: A body with temperature of 100 °F is placed at time t = 0 in a medium maintained at a temperature of 20 °F. If, at the end of 10 minutes the temperature of the body is 50 °F, when will the body be 30 °F??//

(a) Let $x(t)$ denote the temperature of the object in °F at time t, in minutes. Then $x' = k(20 - x)$, $x(0) = 100$, and $x(10) = 50$.

(b) The differential equation may be viewed as separable or linear. Consequently, you may use the techniques from Chapter 2 to deal with it and the initial condition. Somewhat amusingly, you may actually solve this and do no integrations at all once you realize that the equation is, in fact, a constant coefficient linear differential equation with an undetermined coefficient driving function.

Here are the details of that. The DE, $x' + kx = 20k$, is linear with a fundamental set for the corresponding homogeneous equation consisting of $\{ e^{-kt} \}$. The UC driving function is $F(t) = 20k$. Using this and the UC set consisting only of $\{ 1 \}$, results in $x_p(t) = 20$. Thus, a general solution to the DE is given by $x(t) = 20 + c_1e^{-kt}$. Thus, using the I.C. $x(0) = 100$ leads to $x(t) = 20 + 80e^{-kt}$.

(c) By using $x(10) = 50$ now, one can obtain $k = -\ln(3/8)/10$. Thus, $x(t) = 20 + 80(3/8)^{t/10}$. Solving $30 = x(t_0)$ yields $t_0 = 10[(\ln(1/8)/\ln(3/8)] = 10[(\ln(8)/\ln(8/3)]$ which turns out to be approximately 21.2, in minutes, of course. Since you don't have access to a log table you aren't expected to obtain the approximation!!]

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$$\int \sin(x)\cdot e^x dx = y$$