Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.* Test #:

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1. (15 pts.) (a) Suppose that f(t) is defined for t > 0. What is the definition of the Laplace transform of f, $\mathcal{L}\{f(t)\}$, in terms of a definite integral??

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 1 & , \quad 3 \geq t > 0 \\ 2 & , \quad t > 3 \ . \end{cases}$$

$$\begin{aligned} \mathcal{L}\{f(t)\}(s) &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^3 f(t)e^{-st}dt + \int_3^\infty f(t)e^{-st}dt \\ &= \int_0^3 1e^{-st}dt + \int_3^\infty 2e^{-st}dt \\ &= \frac{1}{s} - \frac{e^{-3s}}{s} + \lim_{R\to\infty}\left(\frac{2e^{-3s}}{s} - \frac{2e^{-Rs}}{s}\right) \\ &= \frac{1}{s} + \frac{e^{-3s}}{s} \quad if \quad s > 0. \end{aligned}$$

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2. (10 pts.) Without evaluating any improper integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $f(t) = 12 \cdot \sin^2(4t) - 12t^7 \cdot e^{4t}$

$\mathcal{L}\{f(t)\}(s) = 6/s - (6s/(s^2 + 64)) + 12(7!)/(s-4)^8$

Note: The needed trig identity is the following:

$$\sin^2(\theta) = (1/2)(1 - \cos(2\theta)).$$

Yes, 12(7!) = 60480, but you don’t have to go this far.

(b) $g(t) = 10 \cdot \sin(t) \cdot t^2$

$\mathcal{L}\{g(t)\}(s) = 10\mathcal{L}\{t(t \cdot \sin(t))\}(s) = (-10)d(\mathcal{L}\{t \cdot \sin(t)\}(s))/ds$

$= (-10)d((2s)/(s^2 + 1)^2)/ds$

$= (20)(3s^2 - 1)/(s^2 + 1)^3$ for s > 0.

3. (15 pts.) Very carefully solve the following initial value problem involving an Euler-Cauchy O.D.E.:

$$x^2y''(x) - xy'(x) = 10\cdot\ln(x)$$

$$y(1) = 0,\ y'(1) = 1$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the IVP above transforms into the following IVP in $w(t)$:

$$w''(t) - 2w'(t) = 10\cdot t$$

$$w(0) = 0,\ w'(0) = 1.$$

The corresponding homogeneous equ.: $w''(t) - 2w'(t) = 0$
The auxiliary equation: $m\cdot(m - 2) = 0$
Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ 1, e^{2t} \}$
The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that $w_p(t) = -(5/2)t^2 - (5/2)t$ is a particular integral. Dealing with the initial conditions now or slightly later finally yields the following: $y(x) = -(7/4) + (7/4)x^2 - (5/2)\ln(x) - (5/2)\ln^2(x)$.

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4. (10 pts.) Transform the given initial value problem into an algebraic equation in $\mathcal{L}\{y\}$ and solve for $\mathcal{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

I.V.P.: $y''(t) - 4y'(t) - 5y(t) = 4\cdot\sin(5t)\cos(5t)$

$y(0) = 2\ ,\ y'(0) = -1$

$$\mathcal{L}\{y\} = (s^2-4s-5)^{-1}[(20)/(s^2 + 100)] + (s^2-4s-5)^{-1}(2s-9)$$

is where you should land after a little algebraic prestidigitation. I have distributed the denominator to add to the confusion. Note: You will have needed $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ along the way. Trig-or-treat.

Bonkers Bonus Noise:

$$\int_0^\infty \sin(4t)\cos(2t)e^{-t}dt = \mathcal{L}\{\sin(4t)\cos(2t)\}(1)$$
$$= (1/2)\mathcal{L}\{\sin(6t)+\sin(2t)\}(1)$$
$$= (1/2)\left[\frac{6}{1+36} + \frac{2}{1+4}\right]$$
$$= (3/37) + (1/5) = 52/185$$

Note: On some tests given in prior semesters there was an additional hint: "If you hold your mouth just right and squint just so, you can evaluate the following improper integral with less than ten pages of work." I guess the hint is required...
d.l.r.

5. (10 pts.)

The equation $x^2 \cdot y'' + x(x + 3)y' - 8y = 0$ has a regular singular point at $x_0 = 0$. Theorems 6.2 and 6.3 imply that there is at least one nontrivial solution of the form

$$y_1(x) = |x|^r \sum_{n=0}^{\infty} c_n x^n$$

and that the series converges for each x satisfying $0 < |x| < R$, for some constant $R > 0$. What can you tell me about the exact value of r for the ODE above? [You need not concern yourself with the values of the c_n's.]

To determine the *r* in question, you need the indicial equation for the ODE at $x_0 = 0$ and its roots. Now the indicial equation is given by $r(r-1) + p_0 r + q_0 = 0$, where

$$p_0 = \lim_{x \to 0} x[x(x+3)/x^2] = 3 \quad \text{and} \quad q_0 = \lim_{x \to 0} x^2[(-8)/x^2] = -8 .$$

Thus, the indicial equation is $r^2 + 2r - 8 = 0$, with roots $r_1 = 2$ and $r_2 = -4$. Consequently, the *r* in question is $r_1 = 2$. [The solution corresponding to r_2 *may* involve a logarithmic term. You may also obtain the indicial equation using Ross's method.]

For problems 6, 7, and 8 suppose that $x_0 = 2\pi$ is a regular singular point of a homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$. Given the indicial equation provided, use all the information available and Theorem 6.3 to say what the solutions look like without attempting to obtain the coefficients of the power series involved.

6. (5 pts.) Indicial equation: $(r - (14/3))(r + (7/3)) = 0$

$$y_1(x) = |x-2\pi|^{14/3}\sum_{n=0}^{\infty} c_n(x-2\pi)^n$$

$$y_2(x) = |x-2\pi|^{-7/3}\sum_{n=0}^{\infty} d_n(x-2\pi)^n + Cy_1(x)\ln|x-2\pi|$$

7. (5 pts.) Indicial equation: $(r - 1)(r - 1) = 0$

$$y_1(x) = |x-2\pi|^{1}\sum_{n=0}^{\infty} c_n(x-2\pi)^n$$

$$y_2(x) = |x-2\pi|^{2}\sum_{n=0}^{\infty} d_n(x-2\pi)^n + y_1(x)\ln|x-2\pi|$$

8. (5 pts.) Indicial equation: $(r + 2\pi)(r - \pi) = 0$

$$y_1(x) = |x-2\pi|^{\pi}\sum_{n=0}^{\infty} c_n(x-2\pi)^n$$

$$y_2(x) = |x-2\pi|^{-2\pi}\sum_{n=0}^{\infty} d_n(x-2\pi)^n$$

9. (5 pts.) Using only the definition of the convolution in terms of the definite integral, compute (f*g)(t) when f(t) = 60t and g(t) = $10 \cdot t^2$

$$
\begin{aligned}
(f*g)(t) &= \int_0^t f(x)g(t-x)dx = \int_0^t 60x\cdot(t-x)^2dx \\
&= \int_0^t 600x(t^2-2tx+x^2)dx \\
&= 50t^4 \text{ \textit{after the fundamental dust settles}.}
\end{aligned}
$$

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10. (5 pts.) Suppose that f and g are functions defined for t > 0 with Laplace transforms given by $\mathfrak{L}\{f(t)\}(s) = s^{-3}$ and $\mathfrak{L}\{g(t)\}(s) = (s^2 + 9)^{-1}$. Obtain the following Laplace transform:

$$
\mathfrak{L}\{(f*g)(t) + 4e^{3t}\cdot f(t)\}(s) = \mathfrak{L}\{f\}(s)\mathfrak{L}\{g\}(s) + 4\cdot\mathfrak{L}\{f\}(s-3)
$$

$$
= \frac{1}{s^3(s^2+9)} + \frac{4}{(s-3)^3}
$$

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11. (10 pts.) If $y(x) = \sum_{n=0}^{\infty} c_n x^n$ is the solution of the initial value problem $y'' - 4xy' = 0$ with $y(0) = 0$ and $y'(0) = 1$, obtain the recurrence formula for the coefficients of y(x). What are the values of $c_0, c_1, \ldots, c_4$?

$$
\begin{aligned}
0 &= -4xy' + y'' = -4x\sum_{n=1}^{\infty} nc_nx^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} \\
&= -\sum_{n=1}^{\infty} 4nc_nx^n + \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n \\
&= 2c_2 + \sum_{n=1}^{\infty}[(n+2)(n+1)c_{n+2} - 4nc_n]x^n \\
&\Rightarrow c_2 = 0, \text{ and for } n \geq 1,\ c_{n+2} = \frac{4nc_n}{(n+2)(n+1)}.
\end{aligned}
$$

So $c_0 = y(0) = 0$, $c_1 = y'(0) = 1$, $c_2 = 0$, $c_3 = 2/3$, and, $c_4 = 0$.

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12. (5 pts.) Compute $\mathfrak{L}^{-1}\{F(s)\}$ when $F(s) = \dfrac{8}{(x-1)(x^2+1)}$.

$$
\mathfrak{L}^{-1}\{F(s)\} = \mathfrak{L}^{-1}\left\{\frac{4}{s-1} - \frac{4s+4}{s^2+1}\right\} = 4e^t - 4\cos(t) - 4\sin(t)
$$

after you have done the requisite partial fraction decomposition. [If you used convolution, we shall be here until next week.]

Bonkers 10 Point Bonus: Elsewhere, evaluate the following improper integral: $\int_0^{\infty}\sin(4t)\cos(2t)e^{-t}dt$ This may be found at the bottom of Page 2 of 4. The key insight is that the integral is the Laplace transform of f(t) = sin(4t)cos(2t) at s = 1.