Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.* Test #:

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1. (15 pts.) (a) Suppose that f(t) is defined for $t > 0$. What is the definition of the Laplace transform of f, $\mathcal{L}\{f(t)\}$, in terms of a definite integral??

$\mathcal{L}\{f(t)\}(s) =$

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 1 , & 3 \geq t > 0 \\ 2 , & t > 3 . \end{cases}$$

$\mathcal{L}\{f(t)\}(s) =$

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2. (10 pts.) Without evaluating any improper integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $f(t) = 12\cdot\sin^2(4t) - 12t^7\cdot e^{4t}$

$\mathcal{L}\{f(t)\}(s) =$

(b) $g(t) = 10\cdot\sin(t)\cdot t^2$

$\mathcal{L}\{g(t)\}(s) =$

3. (15 pts.) Very carefully solve the following initial value problem involving an Euler-Cauchy O.D.E.:

$$x^2y''(x) - xy'(x) = 10\cdot\ln(x)$$

$$y(1) = 0,\ y'(1) = 1$$

4. (10 pts.) Transform the given initial value problem into an algebraic equation in $\mathcal{L}\{y\}$ and solve for $\mathcal{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

I.V.P.: $y''(t) - 4y'(t) - 5y(t) = 4\cdot\sin(5t)\cos(5t)$

$$y(0) = 2\ ,\ y'(0) = -1$$

5. (10 pts.)

The equation $x^2 \cdot y'' + x(x + 3)y' - 8y = 0$ has a regular singular point at $x_0 = 0$. Theorems 6.2 and 6.3 imply that there is at least one nontrivial solution of the form

$$y_1(x) = |x|^r \sum_{n=0}^{\infty} c_n x^n$$

and that the series converges for each x satisfying $0 < |x| < R$, for some constant $R > 0$. What can you tell me about the exact value of r for the ODE above? [You need not concern yourself with the values of the c_n's.]

For problems 6, 7, and 8 suppose that $x_0 = 2\pi$ is a regular singular point of a homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$. Given the indicial equation provided, use all the information available and Theorem 6.3 to say what the solutions look like without attempting to obtain the coefficients of the power series involved.

6. (5 pts.) Indicial equation: $(r - (14/3))(r + (7/3)) = 0$

$y_1(x) =$

$y_2(x) =$

7. (5 pts.) Indicial equation: $(r - 1)(r - 1) = 0$

$y_1(x) =$

$y_2(x) =$

8. (5 pts.) Indicial equation: $(r + 2\pi)(r - \pi) = 0$

$y_1(x) =$

$y_2(x) =$

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9. (5 pts.) Using only the definition of the convolution in terms of the definite integral, compute $(f*g)(t)$ when $f(t) = 60t$ and $g(t) = 10\cdot t^2$

$(f*g)(t) =$

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10. (5 pts.) Suppose that f and g are functions defined for $t > 0$ with Laplace transforms given by $\mathcal{L}\{f(t)\}(s) = s^{-3}$ and $\mathcal{L}\{g(t)\}(s) = (s^2 + 9)^{-1}$. Obtain the following Laplace transform:

$\mathcal{L}\{(f*g)(t) + 4e^{3t}\cdot f(t)\}(s) =$

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11. (10 pts.) If $y(x) = \sum_{n=0}^{\infty} c_n x^n$ is the solution of the initial value problem $y'' - 4xy' = 0$ with $y(0) = 0$ and $y'(0) = 1$, obtain the recurrence formula for the coefficients of $y(x)$. What are the values of $c_0, c_1, \ldots, c_4$?

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12. (5 pts.) Compute $\mathcal{L}^{-1}\{F(s)\}$ when $F(s) = \dfrac{8}{(s-1)(s^2+1)}$.

$\mathcal{L}^{-1}\{F(s)\} =$

Bonkers 10 Point Bonus: Elsewhere evaluate the following improper integral: $\int_0^{\infty} \sin(4t)\cos(2t)e^{-t}dt$