1. (40 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $h(t) = \begin{cases} -5 & , 0 < t < 5 \\ 25 & , 5 < t < 15 \\ 2 & , 15 < t . \end{cases} = -5 + 30u_5(t) - 23u_{15}(t)$

$\mathcal{L}\{h(t)\}(s) = -5\mathcal{L}\{u_0(t)\}(s) + 30\mathcal{L}\{u_5(t)\}(s) - 23\mathcal{L}\{u_{15}(t)\}(s)$

$= -(1/s)(5 - 30e^{-5s} + 23e^{-15s})$

or equivalent.

(b) $g(t) = \pi \cdot t \cdot e^{2t} \cdot \cos(t)$

$\mathcal{L}\{g(t)\}(s) = \pi\mathcal{L}\{e^{2t}(t \cdot \cos(t))\}(s)$

$= \pi\mathcal{L}\{t \cdot \cos(t)\}(s - 2)$

$= \pi((s - 2)^2 - 1^2)/[((s - 2)^2 + 1^2)^2]$

$= \pi((s - 2)^2 - 1)/[((s - 2)^2 + 1)^2]$

This can also be handled by following the line of transformation that begins $\mathcal{L}\{g(t)\}(s) = \pi\mathcal{L}\{t(e^{2t} \cdot \cos(t))\}(s)$ and uses differentiation, but it's much messier.

(c) $(f*g)(t)$, when $f(t) = 2 \cdot \sin(3t)$ and $g(t) = 2 \cdot e^{-5t} \cdot t^4$

$\mathcal{L}\{(f*g)(t)\}(s) = \mathcal{L}\{f(t)\}(s) \cdot \mathcal{L}\{g(t)\}(s)$

$= 2\mathcal{L}\{\sin(3t)\}(s) \cdot 2\mathcal{L}\{e^{-5t} \cdot t^4\}(s)$

$= 2 \cdot 3(s^2 + 9)^{-1} \cdot 2 \cdot 4!(s + 5)^{-5}$

$= 288(s^2 + 9)^{-1}(s + 5)^{-5}$ or equivalent.

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10 Point Bonus: If $\mathcal{L}\{f(t)\}(s) = \dfrac{1/s}{1 + e^{-s}}$ *for* $s > 0$, what's f(t), assuming the Laplace beast coexists with the series shaman?

$\mathcal{L}\{f(t)\}(s) = \frac{1}{s}\sum_{k=0}^{\infty}((-1)e^{-s})^k = \sum_{k=0}^{\infty}\frac{(-1)^k e^{-ks}}{s}$ *for* $s > 0$, thanks to the geometric series genie. [Yes, the series converges for $s > 0$.] Observe that one may then realize that

$f(t) = \sum_{k=0}^{\infty}(-1)^k u_k(t)$ *for* $t \geq 0$, where, in fact, the sum is finite for each $t \geq 0$. Examining f(t) over a couple of intervals will reveal that f is 2 - periodic with $f(t) = 1$ for $0 < t < 1$ and $f(t) = 0$ for $1 < t < 2$, and on and on. Really??

1.

(d) $f(t) = \begin{cases} 16 & , 0 < t < 2 \\ 8t & , 2 < t \end{cases} \quad = \quad 16 + (8t - 16)\cdot u_2(t)$

$$\begin{aligned}
\mathfrak{L}\{f(t)\}(s) &= \mathfrak{L}\{16\}(s) + \mathfrak{L}\{(8t - 16)\cdot u_2(t)\}(s) \\
&= 16s^{-1} + \mathfrak{L}\{g(t-2)\cdot u_2(t)\}(s),\ \text{where } g(t-2) = 8t - 16 \\
&= 16s^{-1} + e^{-2s}\mathfrak{L}\{g(t)\}(s),\ \text{with } g(t) = 8(t+2) - 16 \\
&= 16s^{-1} + e^{-2s}\mathfrak{L}\{8t\}(s) \\
&= 16s^{-1} + 8e^{-2s}s^{-2}
\end{aligned}$$

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2. (10 pts.) (a) The Laplace transform of the following 2-periodic function may be written in terms of a definite integral. Simply express the transform in terms of the appropriate definite integral, but do not attempt to evaluate that definite integral.

$$f(t) = \begin{cases} t & , \text{ for } 0 \le t < 1 \\ 2 - t & , \text{ for } 1 \le t < 2, \end{cases}$$

and $f(t) = f(t + 2)$ for $t \ge 0$.

$$\mathfrak{L}\{f(t)\}(s) = \frac{\int_0^2 f(t)e^{-st}dt}{1 - e^{-2s}}$$

(b) The following sum of definite integrals can be realized as the Laplace transform of a certain function g(t) defined for $t \ge 0$. Provide the precise definition of that function g.

$$\int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt = \mathfrak{L}\{g(t)\}(s), \quad \text{where}$$

$$g(t) = \begin{cases} t & , \text{ for } 0 \le t < 1 \\ 2 - t & , \text{ for } 1 \le t < 2, \\ 0 & , \text{ for } 2 \le t. \end{cases}$$

Warning: *Do not attempt to evaluate the Laplace transform of g.*

NOTE: Of course you could also have answered that

$$g(t) = \begin{cases} f(t) & , \text{ for } 0 \le t < 2 \\ 0 & , \text{ for } 2 \le t, \end{cases}$$ where f(t) is defined above.

3. (15 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathcal{L}\{y(t)\}(s) = \frac{se^{-\pi s}}{s^2 + 25} + \frac{10 \cdot s}{(s-2)^2 + 36} .$$

What's the solution, y(t) , to the IVP??

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{se^{-\pi s}}{s^2 + 25} \right\}(t) + 10\mathcal{L}^{-1}\left\{ \frac{s}{(s-2)^2 + 36} \right\}(t)$$

$$= u_\pi(t)\cos(5(t-\pi)) + 10e^{2t}\cos(6t) + (10/3)e^{2t}\sin(6t)$$

after just a little of the usual prestidigitation --- factoring unity correctly and invoking the avatar of zero who transmogrifies ugly toads to princely table forms --- routine magic now. You may write y(t) in piecewise-defined form if you are feeling rowdy.

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4. (15 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y' = f(t) \text{ , where } f(t) = \begin{cases} 6 & \text{, for } 0 \le t < 3 \\ 2t & \text{, for } 3 \le t \end{cases}$$

and $y(0) = -3$.

Observe that $f(t) = 6 + 2(t - 3)u_3(t)$. Thus, applying our friendly Laplace transform to both sides of the differential equation, using the initial condition, and solving for the Laplace transform of y yields

$$\mathcal{L}\{y(t)\}(s) = [6/(s^2)] + [2e^{-3s}/(s^3)] - [3/s] .$$

Thus, after not bowing at all to the partial fraction proprietor, you may write

$$y(t) = 6t + u_3(t)(t-3)^2 - 3$$

Finally, after you march up and down the unit steps a few times, you have

$$y(t) = \begin{cases} 6t - 3 & \text{, for } 0 \le t < 3 \\ t^2 + 6 & \text{, for } 3 \le t \end{cases}$$

more or less. There are, of course, a couple of inequalities that we have fudged. [Look at Test 1, Problem 1(f)...!!??]

5. (10 pts.) Very neatly transform the given initial value problem into a linear system in $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ and stop. Do not attempt to solve for $\mathcal{L}\{x\}$ or $\mathcal{L}\{y\}$.

I.V.P.: $2x'(t) + y(t) = 12t^2$

$y'(t) - 3x(t) = 8 \cdot e^{-4t}\delta(t - 5)$, $x(0) = 1$, $y(0) = -2$

After performing the transformation two-step and tidying things up a mite, you might waltz right up to

$$2s\mathcal{L}\{x\} + \mathcal{L}\{y\} = [24/s^3] + 2$$

$$-3\mathcal{L}\{x\} + s\mathcal{L}\{y\} = 8e^{-5(s+4)} - 2$$

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6. (10 pts.) Consider the following initial value problem:

$$y'(t) - 3y(t) = h(t) \text{ and } y(0) = 0,$$

$$\text{where } h(t) = \begin{cases} 10\sin(t), & \text{for } 0 < t < 2\pi \\ 0 & \text{, for } t > 2\pi. \end{cases}$$

Suppose that

$$y(t) = \begin{cases} e^{3t} - \cos(t) - 3\sin(t) & \text{, for } 0 \le t < 2\pi \\ e^{3t} & \text{, for } t \ge 2\pi. \end{cases}$$

(a) Show that y satisfies the ODE when $0 < t < 2\pi$.

If $0 < t < 2\pi$, then

$$y'(t) - 3y(t) = [3e^{3t}+\sin(t)-3\cos(t)] - 3[e^{3t}-\cos(t)-3\sin(t)]$$
$$= \sin(t) + 9\sin(t) = 10\sin(t).$$

(b) Show that y satisfies the ODE when $t > 2\pi$.

If $t > 2\pi$, then $y'(t) - 3y(t) = [3e^{3t}] - 3[e^{3t}] = 0$.

(c) Show that y satisfies the initial condition.

$y(0) = e^{3(0)}-\cos(0)-3\sin(0) = 1 - 1 - 0 = 0$. [Dead Equine?]

(d) Why wouldn't you want to call y *the solution* to the IVP?

Since $\lim_{t\to 2\pi^-} y(t) = e^{6\pi}-1$ and $\lim_{t\to 2\pi^+} y(t) = e^{6\pi}$, y(t) is discontinuous at $t = 2\pi$. Consequently, y(t) above is not differentiable at $t = 2\pi$, a very desirable property for *the solution* to the IVP to have. [Note: Continuity is necessary for differentiability, not sufficient. You really should check for differentiability, but if the varmint is not continuous, ...]

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10 Point Bonus: If $\mathcal{L}\{f(t)\}(s) = \dfrac{1/s}{1 + e^{-s}}$ *for* $s > 0$, what's f(t), assuming the Laplace beast coexists with the series shaman? [Look on Page 1 of 4??]