Student Number: **Exam Number:**

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all your magic on the page. Eschew obfuscation.*

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1. (140 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2+1}$; $y(1) = \ln(2)$.

(b) $\frac{dy}{dx} = 5x^4(1 - y^2)^{1/2}$; $y(0) = -\frac{1}{2}$.

1.(c) $(x^2 + xy + y^2)dx - (x^2)dy = 0$

(d) $(\sec(x)\tan(x) - 8y^2)dx + (e^{2y} - 16xy)dy = 0$

1.(e) $6\frac{dy}{dx} + \frac{1}{x}y = 14x^2y^{-5}$

(f) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 6\ln(x) + 6$

1.(g) $\frac{d^2y}{dx^2} + y = \tan(x)$; $y(\frac{\pi}{4}) = 0$, $y'(\frac{\pi}{4}) = 0$.

2. (10 pts.) Work the following problem which uses Hooke's law:

Be sure to state what your variables represent using complete sentences.

// A 256 pound stone is attached to the lower end of a spring with a fixed support. (The spring is vertical.) The weight stretches the spring 2 feet when in its equilibrium position. If the weight is then pushed down 8 inches and released at time t = 0 with an initial velocity of 4 inches per second directed upward, obtain the displacement s a function of time. [Assume free, undamped motion.]

3. (10 pts.) Obtain the recurrence formula for the power series solution at $x_0 = 0$ of the homogeneous O.D.E.

$$\frac{d^2y}{dx^2} - x^2y = 0.$$

Are there are any coefficients which must be zero ? Which are they ?? Compute the first six (6) coefficients for the initial conditions y(0) = 2 and y′(0) = 0.

4. (10 pts.)
The equation $x^2 \cdot y'' + x(x + 5)y' + 4y = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved. [Hint: Use ALL the information you have available after solving the indicial equation. Write those power series varmints right carefully, folks.]

5. (10 pts.) It is known that f(x) = x^r is a solution of the homogeneous linear differential equation

(*) $x^2y'' - 3xy' + 4y = 0$

for a particular value of r.

(a) Find the value of r by substituting f(x) into (*), obtaining an algebraic equation in r, and solving the equation involving r.
(b) Then find a second, linearly independent solution to (*) by using the technique of reduction of order.
(c) Using the wronskian, verify that the two functions you get from parts (a) and (b) are indeed linearly independent.

6. (10 pts.) Work the following problem which uses *Newton's Law of Cooling*:

// Assume a body of temperature 200°F is placed at time t = 0 in a medium the temperature of which is maintained at 80°F. At the end of ten minutes the temperature of the body has dropped to 180°F. When will the temperature of the body be 100°F?? //

Be sure to state what your variables represent using complete sentences.

7. (10 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$\begin{cases} x'(t) + y(t) = 0 \\ x(t) - y'(t) = 0\ , \end{cases}$$

$$x(0) = 1 \ \ \textit{and} \ \ y(0) = 0.$$

Can you avoid going in circles??

Silly 10 point bonus: Each polynomial with real coefficients,

$$\begin{aligned} p(x) &= \sum_{k=0}^{n} b_k x^k \\ &= b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1} x^{n-1} + b_n x^n\ , \end{aligned}$$

is the unique solution to infinitely many initial-value problems where the ordinary differential equations have constant coefficients. Obtain an infinite sequence of initial-value problems where each ordinary differential equation is homogeneous with constant coefficients, the initial conditions are at $x_0 = 0$, and the general polynomial, p, above is the unique solution. [Say where your work is, for it won't fit here. Oh, here's a hint: There is an easy, taylor-made solution.]