General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. What is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

1. (75 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [15 points/part]

(a) $\frac{dy}{dx} - \frac{2}{x}y = 3x^4$; $y(1) = 8$

The ODE of problem (a) is linear as written with an obvious integrating factor of $\mu = x^{-2}$ for $x > 0$. Multiplying the ODE by μ, integrating, and then dealing with the initial condition allows you to produce $x^{-2}y = x^3 + 7$, an implicit solution. An obvious explicit solution is $y(x) = x^5 + 7x^2$, defined for $x > 0$.

(b) $(2xy + 1)dx + (x^2 + 4y)dy = 0$

This varmint is plainly exact. A one-parameter family of solutions is given by $x^2y + x + 2y^2 = C$, where C is an arbitrary constant. Explicit solutions should be cheap thrills here. Why?

(c) $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$ *with* $x > 0$.

This is clearly a Bernoulli equation. Turn it into a linear equation using the substitution $v = y^{-1}$. Yadda, yadda, yadda. $y^{-1} = 1 + cx^{-1}$, or more explicitly, $y(x) = (1 + cx^{-1})^{-1}$.

(d) $(x + 2y)dx - (2x + y)dy = 0$

This is a homogeneous equation, with each coefficient function homogeneous of degree 1. First write the equation in the form of $dy/dx = g(y/x)$ by doing suitable algebra carefully. Then by setting $y = vx$, substituting, and doing a bit more algebra, you will end up looking at the separable equation

$$(v^2 - 1)dx + x(v + 2)dv = 0.$$

Separating variables and integrating leads you to

$$\int \frac{1}{x}\,dx + \int \frac{v+2}{v^2-1}dv = C\ .$$

To deal with the second integral, do a partial fraction decomposition. After doing that and integrating, you'll obtain

$$\ln|x| - \frac{1}{2}\ln|v+1| + \frac{3}{2}\ln|v-1| = C\ .$$

You may then finish this by replacing v above with y/x. If you are feeling feisty, clean up the loggy mess. [You don't have time during the exam!]

(e) $\sec(2y)\,dx + 20(1-x^2)^{1/2}dy = 0$; $y(0) = 2\pi$

This is separable as written. Look. Because the ODE in (e) is equivalent to

$$\sec(2y) + 20(1-x^2)^{1/2}\frac{dy}{dx} = 0 ,$$

there are no constant solutions since $\sec(2y)$ has no zeros. By separating variables, cleaning up the algebra, and integrating you get

$$\int \frac{1}{(1-x^2)^{1/2}}\,dx + \int 20\cos(2y)\,dy = C .$$

Thus, a 1-parameter family of solutions is given by means of the equation $\sin^{-1}(x) + 10\cdot\sin(2y) = C$. Applying the initial condition implies that C = 0. Consequently, an implicit solution to the IVP is $\sin^{-1}(x) + 10\cdot\sin(2y) = 0$. Explicit solutions are easy to obtain from this.

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2. (10 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain.**

$$(x - 2y + 1)dx - (4x - 3y - 6)dy = 0$$

Solving the linear system of equations consisting of

$$h - 2k + 1 = 0 \text{ and } 4h - 3k - 6 = 0$$

yields $(h,k) = (2,3)$. The substitution that does the job is evidently, $x = X + 2$ and $y = Y + 3$. When you do the substitution, the original ODE reduces to

$$(X - 2Y)dX - (4X - 3Y)dY = 0 ,$$

an ODE that is homogeneous of degree 1.

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3. (15 pts.) Solve the following first order initial value problem:

$$y'(x) + y(x) = f(x) \text{ and } y(0) = 0,$$

where

$$f(x) = \begin{cases} 2 , & if \quad 0 \le x < 1 \\ 0 , & if \quad 1 \le x. \end{cases}$$

Linear ... with an integrating factor $\mu = e^x$ ... ugh; so gluing the pieces together, we get

$$y(x) = \begin{cases} 2-2e^{-x} , & if \quad 0 \le x < 1 \\ 2(e-1)e^{-x} , & if \quad 1 \le x. \end{cases}$$

[See The TestTomb, Spring 2004 for details of how this is done in a similar problem.]

Bonkers 10 Point Bonus: (a) The Fundamental Theorem of Calculus provides a neat formal solution involving a definite integral with respect to the variable 't' to the following dinky IVP:

$$y'(x) = \cos(x^2) \text{ and } y(0) = 1.$$

What is that solution? (b) Unfortunately $g(x) = \cos(x^2)$ cannot be integrated in elementary terms. Use the answer to (a), the Maclaurin series for cos(x), and term-wise integration, to obtain a power series solution to the IVP. [Say where your work is! You don't have room here!]

(a)

$$y(x) = 1 + \int_0^x \cos(t^2)\,dt \textit{ for all } x.$$

(b)

$$\begin{aligned} y(x) &= 1 + \int_0^x \cos(t^2)\,dt \\ &= 1 + \int_0^x \sum_{k=0}^{\infty} \left[\frac{(-1)^k (t^2)^{2k}}{(2k)!}\right] dt \\ &= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k (t^2)^{2k}}{(2k)!}\,dt \\ &= 1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_0^x t^{4k}\,dt \\ &= 1 + \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)(2k)!} \textit{ for all } x. \end{aligned}$$