General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (15 pts.) Write down the general solution to each of the following linear constant coefficient homogeneous equations.

(a) $y''(x) + 4y'(x) + 4y(x) = 0$

Auxiliary Equation: $m^2 + 4m + 4 = (m + 2)^2 = 0$

Roots of A.E.: $m = -2$ *with multiplicity* 2.

General Solution: $y = c_1 e^{-2x} + c_2 x e^{-2x}$

(b) $y''(x) + 3y'(x) - 10y(x) = 0$

Auxiliary Equation: $m^2 + 3m - 10 = (m + 5)(m - 2) = 0$

Roots of A.E.: $m = -5$ *or* $m = 2$

General Solution: $y = c_1 e^{-5x} + c_2 e^{2x}$

(c) $\dfrac{d^5y}{dx^5} - 4\dfrac{d^3y}{dx^3} = 0$

Auxiliary Equation: $m^5 - 4m^3 = m^3(m + 2)(m - 2) = 0$

Roots of A.E.: $m = 0$ *with multiplicity* 3 *or* $m = 2$ *or* $m = -2$

General Solution: $y = c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 e^{-2x}$

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2. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$(m - \pi)^3(m - (1+2i))^2(m - (1-2i))^2 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**] (b) (5 pt.) What is the order of the differential equation?

$$y = c_1 e^{\pi x} + c_2 x e^{\pi x} + c_3 x^2 e^{\pi x} + c_4 e^{x}\sin(2x) + c_5 e^{x}\cos(2x) + c_6 x e^{x}\sin(2x) + c_7 x e^{x}\cos(2x)$$

The order of the differential equation is 7.

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3. (10 pts.) It turns out that the nonzero function $f(x) = e^x$ is a solution to the homogeneous linear O.D.E.

$$y'' - y = 0.$$

Using only the method of reduction of order, show how to obtain a second, linearly independent solution to this equation.
[**WARNING: No reduction, no credit!! Show all steps of this neatly while using notation correctly. You are being graded on the journey, not the destination.**]

Substitution of $y = ve^x$ into the equation and doing a little algebra yields $0 = v'' + 2v'$. [Observe that you can remove the common exponential varmint!] Letting $w = v'$, and performing the obvious substitution yields $w' + 2w = 0$, a linear homogeneous first order ODE with the integrating factor $\mu = e^{2x}$. By using this appropriately, we get $w = ce^{-2x}$. Thus $v = -(c/2)e^{-2x} + d$. Consequently, by setting $c = -2$ and $d = 0$, we obtain $y = e^{-x}$ ... no surprise, this. In fact, as long as $c \neq 0$, you will obtain a solution with f and y linearly independent. [Go compute the Wronskian!!]

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4. (15 pts.) Very carefully obtain the solution to the initial value problem that follows.

$$(*)\begin{cases} x^2y''(x) + 4xy'(x) + 2y = 4\ln(x)\ ; \\ \\ y(1) = 1\ ,\ \ y'(1) = -1. \end{cases}$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) + 3w'(t) + 2w(t) = 4t.$$

The corresponding homogeneous equ.: $w''(t) + 3w'(t) + 2w(t) = 0.$

The auxiliary equation: $(m + 2)(m + 1) = 0$

Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ e^{-2t}, e^{-t} \}$

The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that

$$w_p(t) = 2t - 3$$

is a particular integral. Consequently, the general solution to the original ODE, the one involving y, is

$$y(x) = c_1x^{-2} + c_2x^{-1} + 2\ln(x) - 3.$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = 5x^{-1} - x^{-2} + 2\ln(x) - 3.$$

5. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - xy' - y = 0.$$

(a) (10 pts.) Obtain the recurrence formula for the coefficients of $y(x)$. (b) (5 pts.) If $y(x)$ also satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$, what is the numerical value of c_4 ??

(a): First,

$$\begin{aligned} 0 &= -y - xy' + y'' \\ &= -\sum_{n=0}^{\infty} c_n x^n - x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= -\sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} (n)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \\ &= (2c_2 - c_0)x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - (n+1)c_n]x^n. \end{aligned}$$

From this you can deduce that $c_2 = c_0/2$, and that for $n \geq 1$, we have

$$c_{n+2} = \frac{c_n}{n+2}.$$

(b): $c_0 = y(0) = 1$, $c_1 = y'(0) = 0$, $c_2 = 1/2$, $c_3 = 0$, *and* $c_4 = 1/8$.

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6. (15 pts.) **Using the method of variation of parameters, not the method of undetermined coefficients**, find a particular integral, y_p, of the differential equation

$$y'' + y = 1$$

[**Hint: Read this problem twice and do exactly what is asked to avoid heartbreak!!** *Do not obtain* y_p *using the method of undetermined coefficients. Do not waste time getting the general solution.*]

Corresponding Homogeneous: $y'' + y = 0$. F.S. = {sin(x), cos(x)}.

If $y_p = v_1\sin(x) + v_2\cos(x)$ then v_1' and v_2' are solutions to the following system:

$$\begin{cases} \sin(x)v_1' + \cos(x)v_2' = 0 \\ \cos(x)v_1' - \sin(x)v_2' = 1 \end{cases}$$

Solving the system yields $v_1' = \cos(x)$ and $v_2' = -\sin(x)$. Thus, by integrating, we obtain $v_1 = \sin(x) + c$ and $v_2 = \cos(x) + d$. Thus,

$$y_p = v_1\sin(x) + v_2\cos(x) = \sin(x)\sin(x) + co(x)\cos(x) = 1.$$

[This is known as becoming one with variation of parameters.]

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7. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' + 4y' + 5y = e^{-2x} + \cos(x)e^{-2x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

First, the corresponding homogeneous equation is

$$y'' + 4y' + 5y = 0.$$

which has an auxiliary equation given by $0 = m^2 + 4m + 5$. Thus a fundamental set of solutions for the corresponding homogeneous equation is $\{ \sin(x)e^{-2x} , \cos(x)e^{-2x} \}$. Taking this into account, we may now write

$$y_p(x) = Ae^{-2x} + Bx\sin(x)e^{-2x} + Cx\cos(x)e^{-2x}$$

or something equivalent. [This was a homework problem.]

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8. (10 pts.) (a) Obtain the differential equation and initial condition needed to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. [For (c), the exact value in terms of natural logs will suffice.]

//A tank initially contains 50 gallons of pure water. Starting at time t = 0, a brine containing 2 pounds of dissolved salt per gallon flows into the tank at a rate of 3 gallons per minute. Suppose the mixture is kept uniformly mixed by constant stirring and flows out of the tank at the same rate at which it enters. When will the tank contain 110 pounds of dissolved salt?//

(a) Let $x(t)$ denote the number of pounds of NaCl in the tank at time t, in minutes. Then $x'(t) = 6 - 3(x(t)/50)$, and $x(0) = 0$.

(b) The differential equation may be viewed as separable or linear. Since we get an explicit solution more readily by treating the varmint as linear, we shall do so.

Here are the details. The DE, $x'(t) + 3(x(t)/50) = 6$, is linear with an integrating factor consisting of $\mu(t) = e^{(3/50)t}$. Thus, performing the usual incantations and deftly passing our writing utensils over the cellulose, we obtain the solution to the I.V.P.:

$$x(t) = 100 - 100e^{-(3/50)t}.$$

(c) The equation $110 = x(t_0)$ is equivalent to

$$e^{-\frac{3}{50}t_0} = -\frac{1}{10}.$$

Since real exponentials are never negative, this equation has no solution and the tank never contains 110 pounds of salt. Observe that $x(t) < 100$ for all t ???

Silly 10 Point Bonus: What linear homogeneous ODE with constant coefficients has a fundamental set of solutions given by $\{ e^x, e^x\sin(x), e^x\cos(x) \}$ Say where your work is!
You may find a brief answer in **de-t2-bo.pdf**.