Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.*

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1. (10 pts.) (a) Suppose that f(t) is defined for $t > 0$. What is the definition of the Laplace transform of f, $\mathfrak{L}\{f(t)\}$, in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 5 \ , & if \ \ 0 < t < 2 \\ 0 \ , & if \ \ 2 < t. \end{cases}$$

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^2 5e^{-st}\,dt + \int_2^R 0e^{-st}\,dt\right]$$

$$= \int_0^2 5e^{-st}\,dt = \begin{cases} 10 \ , & if\ s = 0 \\ \dfrac{5}{s} - \dfrac{5e^{-2s}}{s} \ , & if\ s \neq 0 \end{cases}$$

Observe that the improper integral converges for every real number s.

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2. (15 pts.) (a) If f(t) and g(t) are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, (f*g)(t)??

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx$$

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute (f*g)(t) when f(t) = e^{2t} and g(t) = e^{3t}.

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t e^{2x}e^{3(t-x)}\,dx$$

$$= e^{3t}\int_0^t e^{-x}\,dx = e^{3t}\left[(-e^{-x})\Big|_0^t\right] = e^{3t}\left[1 - e^{-t}\right] = e^{3t} - e^{2t}.$$

(c) Using the Laplace transform table, compute the Laplace transform of f*g when f(t) = $t\cdot\sin(t)$ and g(t) = t^2e^{-t}. [Do not attempt to simplify the algebra after computing the transform.]

$$\begin{aligned}\mathfrak{L}\{(f*g)(t)\}(s) &= \mathfrak{L}\{f(t)\}(s)\mathfrak{L}\{g(t)\}(s) \\ &= \mathfrak{L}\{t\sin(t)\}(s)\mathfrak{L}\{t^2e^{-t}\}(s) \\ &= \left[\frac{2s}{(s^2+1)^2}\right]\cdot\left[\frac{2!}{(s+1)^3}\right]\end{aligned}$$

3. (25 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $$f(t) = \begin{cases} 2\ , & if\ \ 0 < t < 1 \\ -5\ , & if\ \ 1 < t < 2 \\ 2\ , & if\ \ 2 < t. \end{cases}$$

Writing f in terms of our friendly unit step functions, we have

$$f(t) = 2u_0(t) + ((-5)-(2))u_1(t) + (2-(-5))u_2(t)$$

$$= 2u_0(t) - 7u_1(t) + 7u_2(t).$$

Thus, from the linearity of the Laplace transform, we have

$$\mathcal{L}\{f(t)\}(s) = \frac{2}{s} - \frac{7}{s}e^{-s} + \frac{7}{s}e^{-2s}$$

(b) $g(t) = 3te^{4t}\sin(t)$

$$\mathcal{L}\{g(t)\}(s) = 3\mathcal{L}\{e^{4t}(t\sin(t))\}(s) = 3\mathcal{L}\{t\sin(t)\}(s-4) = \frac{3[2(s-4)]}{((s-4)^2+1)^2}.$$

This transform may also be obtained by following a line of reasoning that begins with

$$\mathcal{L}\{g(t)\}(s) = 3\mathcal{L}\{t(e^{4t}\sin(t))\}(s) = -3\frac{d}{ds}\mathcal{L}\{e^{4t}\sin(t)\}(s) = \ldots.$$

Obviously, the second route is slightly messier.

(c) $h(t) = \cos^2(t)$

$$\mathcal{L}\{h(t)\}(s) = \mathcal{L}\{\cos^2(t)\}(s) = \mathcal{L}\left\{\frac{1+\cos(2t)}{2}\right\}(s) = \frac{1}{2s} + \frac{s}{2(s^2+4)}.$$

(d) $f(t) = 8\cdot\delta(t - 10)$

$$\mathcal{L}\{f(t)\}(s) = 8\mathcal{L}\{\delta(t-10)\} = 8e^{-10s}.$$

(e) $$g(t) = \begin{cases} 2t\ , & if\ \ 0 < t < 2 \\ 8\ , & if\ \ 2 < t. \end{cases} = 2t + (8 - 2t)u_2(t)$$

$$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{2t\}(s) + \mathcal{L}\{(8-2t)u_2(t)\}(s)$$

$$= \frac{2}{s^2} + \mathcal{L}\{h(t-2)u_2(t)\}(s), \text{ where } h(t-2) = 8-2t$$

$$= \frac{2}{s^2} + e^{-2s}\mathcal{L}\{h(t)\}, \text{ where } h(t) = h((t+2)-2) = 4-2t$$

$$= \frac{2}{s^2} + e^{-2s}\mathcal{L}\{4-2t\}(s) = \frac{2}{s^2} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^2}\right)$$

4. (5 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^5-4x^3)y''+x^2y'+(x-2)y = 0$$

An equivalent equation in standard form is

$$y'' + \frac{x^2}{x^3(x+2)(x-2)}y' + \frac{x-2}{x^3(x+2)(x-2)}y = 0.$$

From this, we can see easily that $x_0 = 0$ is an irregular singular point of the the equation, and $x_0 = -2$ and $x_0 = 2$ are regular singular points. All other real numbers are ordinary points of the equation.

5. (5 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y''+xy'+(x^2-2)y = 0$$

Obtain the indicial equation at $x_0 = 0$, and determine its roots.

The indicial equation is $r(r-1) + p_0 r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{x}{x^2}\right] = 1 \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{x^2-2}{x^2}\right] = -2.$$

So the indicial equation is $r^2 - 2 = 0$ with roots $2^{1/2}$ and $-2^{1/2}$.

6. (15 pts.) For parts (a), (b), and (c) below pretend that $x_0 = 1$ is a regular singular point for some homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$, with the ODE actually being different for each part. For each part, given the indicial equation provided, use all the information available and Theorem 6.3 to say what the two nontrivial linearly independent solutions look like without attempting to obtain the coefficients of the power series involved.

(a) Indicial equation: $(r - \pi)(r - \pi) = 0$; $r_1 = \pi$ and $r_2 = \pi$

$$y_1(x) = |x-1|^{\pi}\sum_{n=0}^{\infty} c_n(x-1)^n$$

$$y_2(x) = |x-1|^{\pi+1}\sum_{n=0}^{\infty} d_n(x-1)^n + y_1(x)\ln|x-1|$$

(b) Indicial equation: $(r - 2\pi)(r - \pi) = 0$; $r_1 = 2\pi$ and $r_2 = \pi$

$$y_1(x) = |x-1|^{2\pi}\sum_{n=0}^{\infty} c_n(x-1)^n$$

$$y_2(x) = |x-1|^{\pi}\sum_{n=0}^{\infty} d_n(x-1)^n$$

(c) Indicial equation: $(r - (\pi + 1))(r - \pi) = 0$; $r_1 = \pi+1$ and $r_2 = \pi$

$$y_1(x) = |x-1|^{\pi+1}\sum_{n=0}^{\infty} c_n(x-1)^n$$

$$y_2(x) = |x-1|^{\pi}\sum_{n=0}^{\infty} d_n(x-1)^n + C\,y_1(x)\ln|x-1|$$

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7. (10 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathcal{L}\{y(t)\}(s) = \frac{se^{-2\pi s}}{s^2+1} + \frac{4s+8}{s^2+2s+5}.$$

What's the solution, $y(t)$, to the IVP??

$$y(t) = u_{2\pi}(t)\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}(t-2\pi) + \mathcal{L}^{-1}\left\{\frac{4s+8}{(s+1)^2+2^2}\right\}(t)$$

$$= u_{2\pi}(t)\cos(t-2\pi) + 4\mathcal{L}^{-1}\left\{\frac{(s+1)+1}{(s+1)^2+2^2}\right\}(t)$$

$$= u_{2\pi}(t)\cos(t)) + 4e^{-t}\cos(2t) + 2e^{-t}\sin(2t).$$

Obviously, you may expand y into a piecewise-defined varmint.

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8. (15 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y'(t) - y(t) = e^t\cos(t) \ ; \ y(0) = 1.$$

By taking the Laplace Transform of both sides of the differential equation, and using the initial condition, we have

$$\mathcal{L}\{y'(t)\}(s) - \mathcal{L}\{y(t)\}(s) = \mathcal{L}\{e^t\cos(t)\}$$

$$\Rightarrow \quad s\mathcal{L}\{y(t)\}(s) - y(0) - \mathcal{L}\{y(t)\}(s) = \frac{s-1}{(s-1)^2+1}$$

$$\Rightarrow \quad (s-1)\mathcal{L}\{y(t)\}(s) = 1 + \frac{s-1}{(s-1)^2+1}$$

$$\Rightarrow \quad \mathcal{L}\{y(t)\}(s) = \frac{1}{s-1} + \frac{1}{(s-1)^2+1}.$$

By taking inverse transforms now, we quickly obtain

$$y(t) = e^t + e^t\sin(t).$$

Silly 10 Point Bonus: If you hold your mouth just right and squint just so, you can evaluate the following improper integral with less than ten pages of work:

$$\int_0^\infty 2t\sin(t)\cos(t)e^{-t}\,dt = \mathcal{L}\{t\sin(2t)\}(1) = \frac{2(2)(1)}{((1)^2+(2)^2)^2} = \frac{4}{25}.$$

Is this the 16% solution?