Student Number: Exam Number:

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all your magic on the page. Eschew obfuscation.*

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1. (140 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 2e^x$; $y(0) = 1,\ y'(0) = -1.$

(b) $\dfrac{dy}{dx} = 5x^4\cos^2(y)$; $y(0) = \dfrac{\pi}{3}.$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(c) $\dfrac{d^2y}{dx^2} + y = \sec^2(x)$

(d) $\left(4e^{2x} + y^2\right)dx + \left(3y^2 + 2xy\right)dy = 0$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(e) $\left(x \sec\left(\frac{y}{x}\right) + y \right) dx - (x)\, dy = 0$; $y(1) = \frac{\pi}{6}$

(f) $\frac{dy}{dx} + \frac{1}{x} y = \frac{2}{x^2+1}$; $y(1) = \ln(4)$.

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(g) $\frac{dy}{dx} + \frac{1}{x}y = x^2y^{-4}$ *for* $x > 0$.

2. (10 pts.) Work the following problem which uses Hooke's law:

Be sure to state what your variables represent using complete sentences.

// A 256 pound stone is attached to the lower end of a spring with a fixed support. (The spring is vertical.) The weight stretches the spring 2 feet when in its equilibrium position. If the weight is then pushed up 8 inches and released at time t = 0 with an initial velocity of 4 inches per second directed downward, obtain the displacement as a function of time. [Assume free, undamped motion.]

3. (10 pts.) (a) If

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

obtain the recurrence formula(s) satisfied by the coefficients of y above if y is a solution at $x_0 = 0$ of the homogeneous ODE

$$y'' - xy' - y = 0.$$

(b) Compute the first five (5) coefficients of the power series solution y_1 that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

4. (10 pts.) The equation

$$2x^2 y'' - xy' + (x - 5)y = 0.$$

has a regular singular point at $x_0 = 0$. Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved. [Hint: Use ALL the information you have available after solving the indicial equation. Write those power series varmints right carefully, folks.]

5. (5 pts.) Suppose $f_1(x)$, $f_2(x)$, and $f_3(x)$ are the three solutions to

$$(*) \quad y''' - y = 0$$

with

$$f_1(0) = 2, \quad f_1'(0) = 0, \quad f_1''(0) = 0$$

and

$$f_2(0) = 0, \quad f_2'(0) = 1, \quad f_2''(0) = 1$$

and

$$f_3(0) = 3, \quad f_3'(0) = -4, \quad f_3''(0) = -4$$

Is this a fundamental set of solutions for (*)? Explain.
[Hint: You do not actually have to reveal the functions' identities to answer this question!!]

6. (10 pts.) Using only the Laplace transform machine, very carefully solve the following first order initial value problem:

$$y'(t) - y(t) = f(t) \quad \textit{and} \quad y(0) = 0$$

where

$$f(t) = \begin{cases} t, & \textit{if} \quad 0 \le t < 3 \\ 3, & \textit{if} \quad 3 \le t. \end{cases}$$

7. (10 pts.) Using only the Laplace transform machine, very carefully solve the following first order initial value problem:

$$\begin{cases} x'(t) + y(t) = \cos(t) \\ -x(t) + y'(t) = \sin(t) \,, \end{cases}$$

$$x(0) = 0 \ \ \textit{and} \ \ y(0) = 0.$$

Can you avoid spiraling out of control??

8. (5 pts.) Merely, obtain the differential equation and any additional equations that the solution must satisfy to solve the following word problem. Do not attempt to solve the differential equation. State what your variables represent using complete sentences.

// A large water tank initially contains 200 gallons of brine in which 15 pounds of salt is dissolved. Starting at time t = 0 minutes, a brine solution containing 4 pounds of salt per gallon flows into the tank at the rate of 3.5 gallons per minute. The mixture is kept uniform by a mixer which stirs it continuously, and the well-stirred mixture leaves the tank at a rate of 4 gallons per minute. How much salt is in the tank at the end of one hour?//

Ten Point Bonkers Bonus: You may attempt at most one (1) of the following two extra-credit problems in order possibly to earn an additional 10 points. Clearly indicate which, if any, you are attempting and where your work may be found. These are not "one-liners". Some explanation is needed for each, and mere scratching will not be considered seriously.

Problem A: Each polynomial with real coefficients,

$$\begin{aligned} p(x) &= \sum_{k=0}^{n} b_k x^k \\ &= b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1} x^{n-1} + b_n x^n , \end{aligned}$$

is the unique solution to infinitely many initial-value problems where the ordinary differential equations are homogeneous with constant coefficients. Obtain an infinite sequence of initial-value problems where each ordinary differential equation is homogeneous with constant coefficients, the initial conditions are at $x_0 = 0$, and the general polynomial, p, above is the unique solution.

Problem B: In a problem in Chapter 1, it was asserted that the initial-value problem

$$(*) \qquad \frac{dy}{dx} = y^{1/3} \; ; \qquad y(0) = 0$$

has infinitely solutions given by

$$y(x) = \begin{cases} 0 , & \textit{if} \quad x \le c \\ [\frac{2}{3}(x - c)]^{3/2} , & \textit{if} \quad x \ge c \end{cases}$$

where $c \ge 0$. Determine which pairs (x_0, y_0) in the xy-plane are such that the IVP (**) below has a unique solution and completely determine the unique solution in this case. Then show that for any other pair, (x_0, y_0), the corresponding IVP has infinitely many solutions.

$$(**) \qquad \frac{dy}{dx} = y^{1/3} \; ; \qquad y(x_0) = y_0$$