
General directions: Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Write complete sentences, and use notation correctly. What is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.

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1. (75 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [15 points/part]

(a) $\left(y\sec^2(x) + \sec(x)\tan(x)\right)dx + (\tan(x) + 2y)dy = 0$

This varmint is plainly exact. A one-parameter family of solutions is given by $y\tan(x) + \sec(x) + y^2 = C$, where C is an arbitrary constant. Explicit solutions should be cheap thrills here. Why? [2.1 #7]

(b) $\dfrac{dx}{dt} + \dfrac{x}{t^2} = \dfrac{1}{t^2}$

The ODE of problem (b) is linear as written with an obvious integrating factor of

$$\mu(t) = e^{-t^{-1}} = e^{-\frac{1}{t}}.$$

An arbitrary explicit solution is one of the form

$$x(t) = 1 + Ce^{\frac{1}{t}}$$

where C is an arbitrary real number. [2.3 #5] [This may be turned into separable, but you probably will lose x(t) = 1 if you're napping.]

(c) $\left(x\tan\left(\dfrac{y}{x}\right) + y\right)dx - x\,dy = 0$

This is a homogeneous equation. The degree of homogeneity is 1. Look at the "y/x" as a hint. Then write the equation in the form of $dy/dx = g(y/x)$ by doing suitable algebra carefully. After setting $y = vx$, substituting, and doing a bit more algebra, you will end up looking at the separable equation

$$\tan(v)dx - (x)dv = 0.$$

Separating variables and integrating leads you to

$$\int \cot(v)\,dv - \int \frac{1}{x}\,dx = C\ .$$

After doing that and integrating, you'll obtain

$$\ln|\sin(v)| - \ln|x| = C$$

or an equivalent beast. You may then finish this by replacing v above with y/x. If you are feeling feisty, clean up the loggy mess. [You don't have time during the exam! 2.2 # 11]

(d) $\dfrac{dy}{dx} + \dfrac{y}{2x} = \dfrac{x}{y^3}\ , \quad y(1) = 2.$

This is clearly a Bernoulli equation. Turn it into a linear equation using the substitution $v = y^4$. Yadda, yadda, yadda. The an implicit solution to the IVP is $x^2y^4 - x^4 = 15$. An explicit solution is easy. [2.3 #25]

(e) $x\sin(y)dx + (x^2+1)\cos(y)dy = 0$

This is separable as written. Look. Because the ODE in (e) is equivalent to

$$x\sin(y) + (x^2+1)\cos(y)\frac{dy}{dx} = 0 ,$$

there are infinitely many constant solutions from the zeros of $\sin(y)$: $y(x) = k\pi$, k any integer. By separating variables, cleaning up the algebra, integrating, and off-loading the kindling you get

$$(x^2+1)\sin^2(y) = C,$$

a 1-parameter family of solutions. [Kindling?? Example 2.9]

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2. (10 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain**.

$$(x - 2y + 1)dx - (4x - 8y - 6)dy = 0$$

The linear system of equations consisting of

$$h - 2k + 1 = 0 \text{ and } -4h + 8k + 6 = 0$$

involves a pair of parallel lines. Thus, an appropriate substitution is $z = x - 2y$. When you do the substitution, the original ODE reduces to

$$(4 - z)dx + (2z - 3)dz = 0 ,$$

an ODE that is plainly separable in x and z.

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3. (15 pts.) Solve the following first order initial value problem:

$$y'(x) + y(x) = f(x) \text{ and } y(0) = 0,$$

where

$$f(x) = \begin{cases} 1 , & if \quad 0 \le x < 2 \\ 0 , & if \quad 2 \le x. \end{cases}$$

Linear ... with an integrating factor $\mu = e^x$ ... ugh; so gluing the pieces together, we get

$$y(x) = \begin{cases} 1-e^{-x} , & if \quad 0 \le x < 2 \\ (e^2-1)e^{-x} , & if \quad 2 \le x. \end{cases}$$

[See The TestTomb, Spring 2004 for details of how this is done in a similar problem.]

Bonkers 10 Point Bonus: Pretend A, B, C, D, E, and F are fixed garden variety real numbers. The ugly looking homogeneous equation

(*) $$\left(Ax^2 + Bxy + Cy^2\right)dx + \left(Dx^2 + Exy + Fy^2\right)dy = 0$$

is actually very easy to solve when it turns out also to be exact. Determine exactly when this happens, with proof. Note well that you are NOT being asked to actually solve the differential equation!!

[Hint: You might want to ask A, B, C, D, E, and F what exact role they might play in this game. Say where your work is! You don't have room hereto explain things!]

First observe that

$$M(x,y) = Ax^2 + Bxy + Cy^2 \text{ and } N(x,y) = Dx^2 + Exy + Fy^2$$

both have continuous first order partial derivatives in the xy-plane. Consequently, the truth of the equation

(**) $$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

at each point (x,y) of the plane is necessary and sufficient for exactness. It turns out that by actually performing the obvious computations, it follows that equation (**) is equivalent to

(***) $$Bx + 2Cy = 2Dx + Ey$$

being true for each (x,y) in the plane. This happens, of course, precisely when

$$B = 2D \text{ and } E = 2C.$$