General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (30 pts.) Obtain the general solution to each of the following linear homogeneous constant coefficient equations.

(a) $y''(x) - 7y'(x) + 10y(x) = 0$

Auxiliary Equation: $m^2 - 7m + 10 = (m-2)(m-5) = 0$

Roots of A.E.: $m = 2 \text{ or } m = 5$

General Solution: $y = c_1 e^{2x} + c_2 e^{5x}$

(b) $y''(x) - 6y'(x) + 9y(x) = 0$

Auxiliary Equation: $m^2 - 6m + 9 = (m-3)^2 = 0$

Roots of A.E.: $m = 3$ *with multiplicity* 2.

General Solution: $y = c_1 e^{3x} + c_2 x e^{3x}$

(c) $\dfrac{d^5y}{dx^5} + 25\dfrac{d^3y}{dx^3} = 0$

Auxiliary Equation: $m^5 + 25m^3 = m^3(m+5i)(m-5i) = 0$

Roots of A.E.: $m = 0$ *with multiplicity* 3, *or* $m = 5i$ *or* $m = -5i$

General Solution: $y = c_1 + c_2 x + c_3 x^2 + c_4 \cos(5x) + c_5 \sin(5x)$

2. (10 pts.) Find the unique solution to the initial value problem

$$y'' + 4y = 6\sin(x)\ ;\ y(\pi/2) = -1\,,\ y'(\pi/2) = 1$$

given that a fundamental set of solutions to the corresponding homogeneous equation is { $\cos(2x)$, $\sin(2x)$ } and a particular integral to the original ODE is

$$y_p(x) = 2\sin(x).$$

Hint: Save time. Use the stuff served on the platter with the cherry on top.

The general solution to the ODE is

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + 2\sin(x).$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = 3\cos(2x) - \frac{1}{2}\sin(2x) + 2\sin(x).$$

3. (10 pts.) It turns out that the nonzero function $f(x) = \sin(x)$ is a solution to the homogeneous linear O.D.E.

$$y'' + y = 0.$$

Using only the method of reduction of order, show how to obtain a second, linearly independent solution to this equation.
[**WARNING: No reduction, no credit!! Show all steps of this neatly while using notation correctly. You are being graded on the journey, not the destination.**]

Substitution of $y = v \cdot \sin(x)$ into the equation and doing a little algebra yields $0 = v'' + 2\cot(x)v'$. Letting $w = v'$, and performing the obvious substitution yields $w' + 2 \cdot \cot(x)w = 0$, a linear homogeneous first order ODE with the integrating factor $\mu = \sin^2(x)$. [This takes a little work with logs.] By using this appropriately, we get $w = c \cdot \csc^2(x)$. Thus $v = -(c)\cot(x) + d$. Consequently, by setting $c = -1$ and $d = 0$, we obtain $y = \cos(x)$ ... no surprise, this. In fact, as long as $c \neq 0$, you will obtain a solution with f and y linearly independent. [Go compute the Wronskian!!]

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4. (10 pts.) Very carefully obtain the general solution for $x > 0$ to the following ODE:

$$x^2y'' - 6y = \ln(x)$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) - w'(t) - 6w(t) = t.$$

The corresponding homogeneous equ.: $w''(t) - w'(t) - 6w(t) = 0.$

The auxiliary equation: $(m - 3)(m + 2) = 0$

Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ e^{3t}, e^{-2t} \}$

The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that

$$w_p(t) = -\frac{1}{6}t + \frac{1}{36}$$

is a particular integral. Consequently, the general solution to the original ODE, the one involving y, is

$$y(x) = c_1x^3 + c_2x^{-2} - \frac{1}{6}\ln(x) + \frac{1}{36}.$$

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5. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$(m - 1)^2(m - (2i))^3(m - (-2i))^3 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**]
(b) (5 pt.) What is the order of the differential equation?

$$y = c_1e^x + c_2xe^x + c_3\sin(2x) + c_4\cos(2x)$$
$$+ c_5x\sin(2x) + c_6x\cos(2x)$$
$$+ c_7x^2\sin(2x) + c_8x^2\cos(2x)$$

The order of the differential equation is 8.

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6. (10 pts.) **Using the method of variation of parameters, not the method of undetermined coefficients**, find a particular integral, y_p, of the differential equation

$$y'' - y = 2e^x$$

Corresponding Homogeneous: $y'' - y = 0$. F.S. = { e^x , e^{-x} }.

If $y_p = v_1e^x + v_2e^{-x}$ then v_1' and v_2' are solutions to the following system:

$$\begin{cases} e^x v_1' + e^{-x} v_2' = 0 \\ e^x v_1' - e^{-x} v_2' = 2e^x \end{cases}$$

Solving the system yields $v_1' = 1$ and $v_2' = -e^{2x}$.
Thus, by integrating, we obtain $v_1 = x + c$ and $v_2 = -(1/2)e^{2x} + d$.
Thus,

$$y_p = v_1e^x + v_2e^{-x} = xe^x - \frac{1}{2}e^{2x}e^{-x} = \left(x - \frac{1}{2}\right)e^x.$$

Silly 10 Point Bonus: Show how to magically solve the following integral equation without integrating --- either by parts or any other way!!

$$(*) \qquad y = \int x^2 e^x \, dx$$

The silly integral equation is equivalent to $y' = x^2e^x$, a linear, constant coefficient linear ODE with a U.C. driving function. The corresponding homogeneous linear ODE is $y' = 0$, which has as a fundamental set of solutions the singleton, { 1 }, which contains the lowly "one function". [The general solution to $y' = 0$ provides the arbitrary *constant of integration*,] Given the fundamental set above and the U.C. driving function, we'd expect to have a particular integral of the form $y_p = Ax^2e^x + Bxe^x + Ce^x$. If y_p is to be a particular integral, then, for each x we must have $Ax^2e^x + (2A + B)xe^x + (B + C)e^x = y_p' = x^2e^x$. From the linear independence of x^2e^x, xe^x, and e^x , this is equivalent to A, B, and C satisfying the following linear system: $A = 1$, $2A + B = 0$, and $B + C = 0$. This is equivalent to $A = 1$, $B = -2$, and $C = 2$. Bottom line:

$$\int x^2e^x \, dx = c \cdot 1 + x^2e^x - 2xe^x + 2e^x$$

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7. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' - 6y' + 8y = x^2 + e^{-2x} + e^{+4x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

First, the corresponding homogeneous equation is

$$y'' - 6y' + 8y = 0.$$

which has an auxiliary equation given by $0 = m^2 - 6m + 8$.
Thus a fundamental set of solutions for the corresponding homogeneous equation is { e^{2x} , e^{4x} }. Taking this into account, we may now write

$$y_p(x) = Ax^2 + Bx + C + De^{-2x} + Exe^{4x}$$

or something equivalent.

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8. (10 pts.) (a) Assuming Newton's Law of Cooling is applicable, obtain the differential equation and any additional equations that the solution must satisfy to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. [For (c), the exact value in terms of natural logs will suffice.]

//A body with temperature of 100 °F is placed at time t = 0 in a medium maintained at a temperature of 40 °F. If, at the end of 10 minutes the temperature of the body is 90 °F, when will the body be 50 °F??//

(a) Let x(t) denote the temperature of the object in °F at time t, in minutes. Then $x' = k(40 - x)$, $x(0) = 100$, and $x(10) = 90$.
(b) The differential equation may be viewed as separable or linear. Consequently, you may use the techniques from Chapter 2 to deal with it and the initial condition. Somewhat amusingly, you may actually solve this and do no integrations at all once you realize that the equation is, in fact, a constant coefficient linear differential equation with an undetermined coefficient driving function.

Here are the details of that. The DE, $x' + kx = 40k$, is linear with a fundamental set for the corresponding homogeneous equation consisting of { e^{-kt} }. The UC driving function is $F(t) = 40k$. Using this and the UC set consisting only of { 1 }, results in $x_p(t) = 40$. Thus, a general solution to the DE is given by $x(t) = 40 + c_1e^{-kt}$. Thus, using the I.C. $x(0) = 100$ leads to $x(t) = 40 + 60e^{-kt}$.

(c) By using $x(10) = 90$ now, one can obtain $k = -\ln(5/6)/10$. Thus, $x(t) = 40 + 60(5/6)^{t/10}$. Solving $50 = x(t_0)$ yields
$t_0 = 10[(\ln(1/6)/\ln(5/6)] = 10[(\ln(6)/\ln(6/5)]$ which turns out to be approximately 98.27, in minutes, of course. [Since you don't have access to a log table, you aren't expected to obtain the approximation!!]

Silly 10 Point Bonus: Show how to magically solve the following integral equation without integrating --- either by parts or any other way!!

$$y = \int x^2 e^x \, dx$$

[Say where your work is, for it won't fit here.] Look on the bottom of Page 3 of 4.