Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.

1. (7 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$((x-4)^2(x-1)^2)y'' + 4x(x-4)y' + (x-1)y = 0$$

An equivalent equation in standard form is

$$y'' + \frac{4x(x-4)}{(x-4)^2(x-1)^2}y' + \frac{x-1}{(x-4)^2(x-1)^2}y = 0.$$

From this, we can see easily that $x_0 = 1$ is an irregular singular point of the the equation, and $x_0 = 4$ is a regular singular point. All other real numbers are ordinary points of the equation.

2. (18 pts.) For parts (a), (b), and (c) below pretend that $x_0 = 4$ is a regular singular point for some homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$, with the ODE actually being different for each part. For each part, given the indicial equation at $x_0 = 4$ provided, use all the information available and Theorem 6.3 to say what the two nontrivial linearly independent solutions look like without attempting to obtain the coefficients of the power series involved.

(a) Indicial equation: (r - (1/5))(r - (1/5)) = 0

$$y_1(x) = |x-4|^{1/5}\sum_{n=0}^{\infty} c_n(x-4)^n$$

$$y_2(x) = |x-4|^{6/5}\sum_{n=0}^{\infty} d_n(x-4)^n + y_1(x)\ln|x-4|$$

(b) Indicial equation: (r + (5/2))(r - (1/2)) = 0

$$y_1(x) = |x-4|^{1/2}\sum_{n=0}^{\infty} c_n(x-4)^n$$

$$y_2(x) = |x-4|^{-5/2}\sum_{n=0}^{\infty} d_n(x-4)^n + C\,y_1(x)\ln|x-4|$$

(c) Indicial equation: (r - (3/2))(r + (1/3)) = 0

$$y_1(x) = |x-4|^{3/2}\sum_{n=0}^{\infty} c_n(x-4)^n$$

$$y_2(x) = |x-4|^{-1/3}\sum_{n=0}^{\infty} d_n(x-1)^n$$

Note: To deal with the bonus problem, you should be able to very quickly derive a trig identity for sin(α)cos(β) real-time from

sin(α+β) = sin(α)cos(β) + sin(β)cos(α)

and

sin(α-β) = sin(α)cos(β) - sin(β)cos(α).

Using it, it follows that 2·sin(3t)cos(t) = sin(4t) + sin(2t)],

3. (15 pts.) (a) If $f(t)$ and $g(t)$ are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, $(f*g)(t)$??

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx$$

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$ when $f(t) = e^{7t}$ and $g(t) = e^{3t}$.

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t e^{7x}e^{3(t-x)}\,dx$$
$$= e^{3t}\int_0^t e^{4x}\,dx = e^{3t}\left[\left(\frac{1}{4}e^{4x}\right)\Bigg|_0^t\right] = \ldots = \frac{e^{7t} - e^{3t}}{4}.$$

(c) Using the Laplace transform table, compute the Laplace transform of $f*g$ when $f(t) = t\cdot\cos(t)$ and $g(t) = t^3e^{-t}$. [Do not attempt to simplify the algebra after computing the transform.]

$$\mathfrak{L}\{(f*g)(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\mathfrak{L}\{g(t)\}(s)$$
$$= \mathfrak{L}\{t\cos(t)\}(s)\cdot\mathfrak{L}\{t^3e^{-t}\}(s)$$
$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]\cdot\left[\frac{3!}{(s+1)^4}\right]$$

4. (10 pts.) (a) Suppose that $f(t)$ is defined for $t > 0$. What is the definition of the Laplace transform of f, $\mathfrak{L}\{f(t)\}$, in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0\ , & if\ \ 0 < t < 2 \\ 4\ , & if\ \ 2 < t. \end{cases}$$

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^2 0e^{-st}\,dt + \int_2^R 4e^{-st}\,dt\right]$$
$$= \lim_{R\to\infty}\left[\frac{4e^{-2s}}{s} - \frac{4e^{-Rs}}{s}\right] = \frac{4e^{-2s}}{s}\ \ provided\ s > 0.$$

Note: You may, of course, check your "answer" using #15 in the table.

Silly 10 Point Bonus: If you hold your mouth just right and squint just so, you can evaluate the following improper integral with less than ten pages of work:

$$\int_0^\infty 2\sin(3t)\cos(t)e^{-t}\,dt = \mathfrak{L}\{2\sin(3t)\cos(t)\}(1)$$
$$= \mathfrak{L}\{\sin(4t)\}(1) + \mathfrak{L}\{\sin(2t)\}(1)$$
$$= \frac{4}{4^2+1^2} + \frac{2}{2^2+1^2} = \frac{4}{17} + \frac{2}{5} = \frac{54}{85}.$$

The trigonometry you might have forgotten may be found on the bottom of page 1. e.t.

5. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' + xy' + y = 0.$$

Obtain the recurrence formula(s) for the coefficients of $y(x)$.

First,

$$\begin{aligned} 0 &= y + xy' + y'' \\ &= \sum_{n=0}^{\infty} c_n x^n + x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= \sum_{n=0}^{\infty} c_n x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \\ &= (c_0 + 2c_2)x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n. \end{aligned}$$

From this you can deduce that $c_2 = -c_0/2$, and that for $n \geq 1$, we have

$$c_{n+2} = -\frac{(n+1)c_n}{(n+2)(n+1)} = -\frac{c_n}{n+2}.$$

6. (10 pts.)

Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{4}{s^5} + \dfrac{3s+4}{s^2+5}$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \frac{4}{4!}t^4 + 3\cos(\sqrt{5}\,t) + \frac{4}{\sqrt{5}}\sin(\sqrt{5}\,t)$$

(b) $F(s) = \dfrac{8s + 7}{(s-3)^2 + 25}$

$$\mathcal{L}^{-1}\{F(s)\}(t) = 8e^{3t}\cos(5t) + \frac{31}{5}e^{3t}\sin(5t)$$

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7. (7 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$3xy'' - (x-2)y' - 2y = 0$$

Theorems 6.2 and 6.3 imply that there is at least one nontrivial solution of the form

$$y_1(x) = |x|^r \sum_{n=0}^{\infty} c_n x^n$$

and that the series converges for each x satisfying $0 < |x| < R$, for some constant $R > 0$. What can you tell me about the exact value of r for the ODE above? [You need not concern yourself with the values of the c_n's.]

To determine r, you need the indicial equation at $x_0 = 0$ and its roots. Now the indicial equation is $r(r-1) + p_0 r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{(-1)(x-2)}{3x}\right] = \frac{2}{3}, \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{-2}{3x}\right] = 0.$$

Thus, the indicial equation is $r^2 - (1/3)r = 0$, with roots $r_1 = 1/3$ and $r_2 = 0$. Consequently, the r in question is $r_1 = 1/3$. [You may also obtain the indicial equation using Ross's method.]

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8. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions that are analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$(n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + c_n = 0 \quad \textit{for all } n \ge 1, \quad \textit{and} \quad c_2 = -\frac{1}{2}c_0.$$

Compute the numerical values of the coefficients c_0, c_1, c_2, c_3, and c_4 for the particular solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$

Here is a more user friendly form of the recursive definition of the c_n's:

$$c_{n+2} = \frac{n(n+1)c_{n+1} - c_n}{(n+2)(n+1)} \quad \textit{for all } n \ge 1, \quad \textit{and} \quad c_2 = -\frac{1}{2}c_0.$$

Clearly, $y(0) = 1$ implies that $c_0 = 1$ and $y'(0) = -1$ implies that $c_1 = -1$. We may now use the known value of c_0 and the second equation above to see that $c_2 = -1/2$. Then using the first equation with n = 1 and n = 2, plus a little arithmetical magic reveals successively that $c_3 = 0$ and $c_4 = 1/24$.

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9. (8 pts.) Transform the given initial value problem into an algebraic equation in $\mathcal{L}\{y\}$ and solve for $\mathcal{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$2y''(t) + 3y'(t) + 4y(t) = e^{5t} \; ; \; y(0) = -3 \; , \; y'(0) = 2 \; .$$

Applying the Laplace transform operator to BOTH SIDES OF THE ODE, using the two initial conditions, and then solving for the transform of y should reveal that

$$\mathcal{L}\{y(t)\}(s) = \left[\frac{1}{2s^2+3s+4}\right]\cdot\left[\frac{1}{s-5} - 6s - 5\right]$$

[A common error: Failure to parenthesize the first and second derivative's transform correctly.]

Silly 10 Point Bonus: If you hold your mouth just right and squint just so, you can evaluate the following improper integral with less than ten pages of work:

$$\int_0^{\infty} 2\sin(3t)\cos(t)e^{-t}\,dt$$

Say where your work is, for it really won't fit here. Look on pages 2 and 1.