
Read Me First: Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals", " \Rightarrow " denotes "implies", and " \Leftrightarrow " denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.

Test #:

1. (7 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$((x-4)^2(x-1)^2)y'' + 4x(x-4)y' + (x-1)y = 0$$

2. (18 pts.) For parts (a), (b), and (c) below pretend that $x_0 = 4$ is a regular singular point for some homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$, with the ODE actually being different for each part. For each part, given the indicial equation at $x_0 = 4$ provided, use all the information available and Theorem 6.3 to say what the two nontrivial linearly independent solutions look like without attempting to obtain the coefficients of the power series involved.

(a) Indicial equation: $(r - (1/5))(r - (1/5)) = 0$

$$y_1(x) =$$

$$y_2(x) =$$

(b) Indicial equation: $(r + (5/2))(r - (1/2)) = 0$

$$y_1(x) =$$

$$y_2(x) =$$

(c) Indicial equation: $(r - (3/2))(r + (1/3)) = 0$

$$y_1(x) =$$

$$y_2(x) =$$

3. (15 pts.) (a) If $f(t)$ and $g(t)$ are piecewise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g , $(f*g)(t)$??

$$(f*g)(t) =$$

(b) Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$ when $f(t) = e^{7t}$ and $g(t) = e^{3t}$.

$$(f*g)(t) =$$

(c) Using the Laplace transform table, compute the Laplace transform of $f*g$ when $f(t) = t \cdot \cos(t)$ and $g(t) = t^3 e^{-t}$. [Do not attempt to simplify the algebra after computing the transform.]

$$\mathcal{L}\{(f*g)(t)\}(s) =$$

4. (10 pts.) (a) Suppose that $f(t)$ is defined for $t > 0$. What is the definition of the Laplace transform of f , $\mathcal{L}\{f(t)\}$, in terms of a definite integral??

$$\mathcal{L}\{f(t)\}(s) =$$

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 & , \text{ if } 0 < t < 2 \\ 4 & , \text{ if } 2 < t. \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) =$$

5. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' + xy' + y = 0.$$

Obtain the recurrence formula(s) for the coefficients of $y(x)$.

6. (10 pts.) Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \frac{4}{s^5} + \frac{3s+4}{s^2+5}$

(b) $F(s) = \frac{8s + 7}{(s-3)^2 + 25}$

7. (7 pts.)

The equation below has a regular singular point at $x_0 = 0$.

$$3xy'' - (x-2)y' - 2y = 0$$

Theorems 6.2 and 6.3 imply that there is at least one nontrivial solution of the form

$$y_1(x) = |x|^r \sum_{n=0}^{\infty} c_n x^n$$

and that the series converges for each x satisfying $0 < |x| < R$, for some constant $R > 0$. What can you tell me about the exact value of r for the ODE above? [You need not concern yourself with the values of the c_n 's.]

8. (10 pts.)

The solution to a certain linear ordinary differential equation with coefficient functions that are analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$(n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + c_n = 0 \text{ for all } n \geq 1, \text{ and } c_2 = -\frac{1}{2}c_0.$$

Determine the coefficients c_0 , c_1 , c_2 , c_3 , and c_4 for the particular solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$

9. (8 pts.)

Transform the given initial value problem into an algebraic equation in $\mathfrak{L}\{y\}$ and solve for $\mathfrak{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$2y''(t) + 3y'(t) + 4y(t) = e^{5t}; \quad y(0) = -3, \quad y'(0) = 2.$$

Silly 10 Point Bonus:

If you hold your mouth just right and squint just so, you can evaluate the following improper integral with less than ten pages of work:

$$\int_0^{\infty} 2 \sin(3t) \cos(t) e^{-t} dt$$

Say where your work is, for it really won't fit here.