Student Number: Exam Number:

Read Me First:

Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Good Luck! [Total Points: 160]

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1. (80 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $\frac{dy}{dx} + 4xy = 8xy^{-3}$; $y(0) = 2$

(b) $\left(x\sec\left(\frac{y}{x}\right) + y\right)dx - x\,dy = 0$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(c) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 8x$ with $x > 0$.

(d) $\dfrac{d^2y}{dx^2} + y = \sec^3(x)$

2. (8 pts.) Obtain the initial-value problem needed to solve each of the following problems. State clearly what your variables represent. *Do not attempt to actually solve the differential equations or initial-value problems you provide.*

(a) // An sixteen pound weight is attached to the lower end of a coil spring suspended from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 9 inches. The weight is then pushed up 8 inches above its equilibrium position and released at t = 0. The medium offers a resistance in pounds numerically equal to 6x′, where x′ is the instantaneous velocity in feet per second. Assuming there are no externally impressed forces, determine the displacement of the weight as a function of time. //

(b) // A large 200 gallon tank initially contains 50 gallons of brine in which there is dissolved 10 pounds of salt. At time $t_0 = 0$, brine containing 2 pounds of dissolved salt per gallon flows into the tank at the rate of 5 gallons/minute. The mixture is kept uniform by stirring, and the well-stirred mixture flows out of the tank at the slower rate of 4 gallons/minute. How much salt is in the tank at the moment that it overflows. //

3. (12 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows. (4 pts./part)

(a)

$$f(t) = \begin{cases} 1 , & if \quad 0 < t < 1 \\ -5 , & if \quad 1 < t < 2 \\ 3 , & if \quad 2 < t. \end{cases} =$$

$\mathfrak{L}\{f(t)\}(s) =$

(b) $h(t) = \sin^2(3t)\, e^{4t} =$

$\mathfrak{L}\{h(t)\}(s) =$

(c)

$$g(t) = \begin{cases} 2t , & if \quad 0 < t < 3 \\ 6 , & if \quad 3 < t. \end{cases} =$$

$\mathfrak{L}\{g(t)\}(s) =$

4. (10 pts.) (a) Obtain the recurrence formula(s) satisfied by the coefficients of the power series solution y at $x_0 = 0$, an ordinary point of the homogeneous ODE,

$$y'' - xy = 0.$$

(b) Compute the first five (5) coefficients of the power series solution y_1 that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$.

5. (10 pts.) The equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

has a regular singular point at $x_0 = 0$. (a) Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. (b) Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved.

6. (18 pts.) Solve the following first order initial-value problem using only the Laplace transform machine.

$$y'(t) - y(t) = 3t^2e^t + 25 \quad ; \; y(0) = 2\pi.$$

7. (12 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathfrak{L}\{y(t)\}(s) = \frac{10se^{-\pi s}}{s^2 + 9} + \frac{8s + 49}{(s+6)^2 + 9}$$

What's the solution to the IVP??

$y(t) =$

8. (10 pts.) (a) Given $f(x) = x^{-1}$ is a nonzero solution to

$$(x^2+x)y'' + (2-x^2)y' - (2+x)y = 0 \ ,$$

obtain a second, linearly independent solution by reduction of order.
(b) Use the Wronskian to prove the two solutions are linearly independent.

Bonkers 10 Point Bonus: You may attempt at most one of the following 3 bonus problems. Clearly indicate which one and where your work is.

(A) Reveal in gory detail how #17, the Laplace Transform of $f(t) = t^{\alpha}$, is obtained.

(B) Obtain the general form of all the coefficients of the first solution, $y_1(x)$, to the differential equation in Problem 5, on Page 4 of 6, in terms of $c_0 = 1$.

(C) In a problem in Chapter 1, it was asserted that the initial-value problem

(*) $$\frac{dy}{dx} = y^{1/3} \ ; \ y(0) = 0$$

has infinitely solutions given by

$$y(x) = \begin{cases} 0 & , \quad if \quad x \le c \\ [\frac{2}{3}(x-c)]^{3/2} & , \quad if \quad x \ge c \end{cases}$$

where $c \ge 0$. Determine which pairs (x_0, y_0) in the xy-plane are such that the IVP (**) below has a unique solution and completely determine the unique solution in this case. Then show that for any other pair, (x_0, y_0), the IVP has infinitely many solutions.

(**) $$\frac{dy}{dx} = y^{1/3} \ ; \ y(x_0) = y_0$$