General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Write complete sentences, and use notation correctly. What is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

1. (80 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [20 points/part]

(a) $\dfrac{dr}{d\theta} + \tan(\theta)r = 4\cos^3(\theta)$ This varmint is linear all day long, and similar to the assigned homework Problems 11 and 23, of Section 2.3. Near $\theta = 0$, an integrating factor is easy to come by:

$$\mu(\theta) = e^{\int \tan(\theta)d\theta} = e^{\ln|\sec(\theta)|} = \sec(\theta)$$

for $\theta \in (-\pi/2,\pi/2)$. Multiplying both sides of the DE by μ results in the following derivative equation:

$$\frac{d}{d\theta}(\sec(\theta)r(\theta)) = 4\cos^2(\theta).$$

The only problem we might encounter is in doing the integration:

$$\begin{aligned}\sec(\theta)r(\theta) = \int \frac{d}{d\theta}(\sec(\theta)r(\theta))\, d\theta &= \int 4\cos^2(\theta)\, d\theta \\ &= \int 2 + 2\cos(2\theta)\, d\theta \\ &= 2\theta + \sin(2\theta) + C.\end{aligned}$$

Obviously that involved a little Trig or Treat, without an identity crisis.
An explicit solution near $\theta = 0$ is given by

$$r(\theta) = (\, 2\theta + \sin(2\theta) + C\,)\cos(\theta).$$

(b) $(4y^2 + xy + x^2)dx - (x^2)dy = 0$

It's easy to see the ODE is homogeneous, and that the degree of homogeneity is 2. Then write the equation in the form of $dy/dx = g(y/x)$ by doing suitable algebra carefully. After setting $y = vx$, substituting, and doing a bit more algebra, you will end up looking at the separable equation

$$x\, dv - (1 + 4v^2)\, dx = 0$$

Separating variables and integrating leads you to

$$\int \frac{1}{1 + 4v^2}\, dv - \int \frac{1}{x}\, dx = C\, .$$

After doing that and evaluating the two integrals, you'll obtain

$$\frac{1}{2}\tan^{-1}(2(\frac{y}{x})) - \ln(x) = C \textit{ for } x > 0\, .$$

An equivalent explicit solution is something like

$$y(x) = \frac{x}{2}\tan(2(\ln(x) + C)) \textit{ for } x > 0.$$

(c) $\left(y^2e^{2x} + \pi\right)dx + \left(ye^{2x} + 2y\right)dy = 0$

Since $\dfrac{\partial}{\partial y}(y^2e^{2x} + \pi) = 2ye^{2x} = \dfrac{\partial}{\partial x}(ye^{2x} + 2y)$, this equation is exact. Thus, there is a nice function $F(x,y)$ satisfying

$$(1)\ \frac{\partial F}{\partial x} = y^2e^{2x} + \pi \quad \textit{and} \quad (2)\ \frac{\partial F}{\partial y} = ye^{2x} + 2y .$$

It follows from (1) above that we have

$$(3)\ F(x,y) = \int y^2e^{2x} + \pi\, dx = \frac{y^2}{2}e^{2x} + \pi x + c(y) ,$$

where $c(y)$ is some function of y whose identity we have yet to determine. Now using (2) and (3) together,

$$ye^{2x} + 2y = \frac{\partial}{\partial y}\left(\frac{y^2}{2}e^{xy} + \pi x + c(y)\right) = ye^{2x} + \frac{dc}{dy}(y) ,$$

which implies $\dfrac{dc}{dy}(y) = 2y$. Integrating yields $c(y) = y^2 + c_0$ for some constant c_0. Thus, $F(x,y) = \dfrac{y^2}{2}e^{2x} + \pi x + y^2 + c_0$. A one-parameter family of implicit solutions is given by

$$\frac{y^2}{2}e^{2x} + \pi x + y^2 = C ,$$

where C is an arbitrary constant.

(d) $5\dfrac{dy}{dx} + \dfrac{y}{x} = 20x^3y^{-4}$ *with* $y(1) = 2$. The equation is obviously a Bernoulli equation. Since the DE is equivalent to the equation

$$5y^4\frac{dy}{dx} + \frac{1}{x}y^5 = 20x^3 ,$$

set $v = y^5$. Then $\dfrac{dv}{dx} = 5y^4\dfrac{dy}{dx}$. Substituting yields the linear equation

$$\frac{dv}{dx} + \frac{1}{x}v = 20x^3 ,$$

which has $\mu = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$ for $x > 0$ as an integrating factor.

Multiplying the linear equation by μ yields $\dfrac{d[xv]}{dx} = 20x^4$. Integrating both sides, we have $xv = 4x^5 + C$. Replacing v results in a one-parameter family of solutions that is given by $xy^5 = 4x^5 + C$. By using the initial condition, $y(1) = 2$, it follows that $C = 28$. Thus, an implicit solution to the IVP is $xy^5 = 4x^5 + 28$. An explicit solution is given by

$$y(x) = (4x^4 + 28x^{-1})^{1/5} \text{ for } x > 0.$$

2. (6 pts.) If the function $f(x) = (x^3 + 4\pi)e^{-3x}$ a solution to the differential equation $\frac{dy}{dx} + 3y = g(x)$, what must the function $g(x)$ be??

Doh!! This, of course, is cheap thrills. If f, above, is a solution to the differential equation, then we must have

$$g(x) = \frac{df}{dx}(x) + 3f(x)$$

$$= \left(3x^2e^{-3x} - 3(x^3 + 4\pi)e^{-3x}\right) + 3(x^3 + 4\pi)e^{-3x}$$

$$= 3x^2e^{-3x}.$$

3. (6 pts.) It is known that every solution to the differential equation $y'' - 4y = 0$ is of the form

$$y(x) = c_1e^{2x} + c_2e^{-2x} .$$

Which of these functions satisfies the initial conditions $y(0) = 2$ and $y'(0) = 8$??

The initial conditions lead to the system of equations

$$\begin{cases} 2 = c_1 + c_2 \\ 8 = 2c_1 - 2c_2 \end{cases} \text{ which is equivalent to } \begin{cases} c_1 = 3 \\ c_2 = -1 \end{cases}$$

The solution to the IVP is given by

$$y(x) = 3e^{2x} - e^{-2x} .$$

4. (8 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain**.

$$(2x + 3y + 1)dx + (4x + 6y + 1)dy = 0$$

The key to this puzzle is the solution to the linear system

$$\begin{cases} 2h + 3k + 1 = 0 \\ 4h + 6k + 1 = 0 \end{cases} \text{ which is equivalent to } \begin{cases} 2h + 3k = -1 \\ 2h + 3k = -\frac{1}{2} \end{cases},$$

geometrically, a pair of parallel lines. Consequently, a suitable substitution is given by $z = 2x + 3y$. After a little routine algebra, the reduction results in the separable DE

$$(1 - z)dx + (2z + 1)dz = 0$$

Bonkers 10 Point Bonus: Give an example of an initial-value problem with a first order ordinary differential equation and infinitely many solutions. Then provide two distinct functions that are solutions and verify that each of the two functions really is a solution to the IVP. [Say where your work is! You don't have room here!]

To get to the root of this, go back and do Problem 8 of Section 1.3. Make sure you understand why Theorem 1.1 is not applicable. Em Toidi.