General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (30 pts.) Obtain the general solution to each of the following linear homogeneous constant coefficient equations.

(a) $y''(x) - 2y'(x) + 5y(x) = 0$

Auxiliary Equation: $0 = m^2-2m+5 = ((m-1)+2i)((m-1)-2i)$

Roots of A.E.: $m = 1+2i$ *or* $m = 1-2i$.

General Solution: $y = c_1e^x\cos(2x) + c_2e^x\sin(2x)$

(b) $y''(x) - 2y'(x) - 8y(x) = 0$

Auxiliary Equation: $0 = m^2-2m-8 = (m+2)(m-4)$

Roots of A.E.: $m = -2$ *or* $m = 4$

General Solution: $y = c_1e^{-2x} + c_2e^{4x}$

(c) $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} = 0$

Auxiliary Equation: $0 = m^4-m^3 = m^3(m-1)$

Roots of A.E.: $m = 0$ *with multiplicity* 3, *or* $m = 1$

General Solution: $y = c_1 + c_2x + c_3x^2 + c_4e^x$

2. (10 pts.) Find the unique solution to the initial value problem

$$y'' - 2y' - 3y = -10\sin(x) \; ; \quad y(0) = 2, \quad y'(0) = 4,$$

given that a fundamental set of solutions to the corresponding homogeneous equation is { e^{3x}, e^{-x} } and a particular integral to the original ODE is

$$y_p(x) = 2\sin(x) - \cos(x).$$

Hint: Save time. Use the stuff served on the platter with the cherry on top.

The general solution to the ODE is

$$y(x) = c_1e^{3x} + c_2e^{-x} + 2\sin(x) - \cos(x).$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = \frac{5}{4}e^{3x} + \frac{7}{4}e^{-x} + 2\sin(x) - \cos(x).$$

3. (15 pts.) Find a particular integral, y_p, of the differential equation

$$y'' + y = \sec^3(x).$$

Obviously the driving function here is NOT a UC function. Thus, we must use variation of parameters to nab the culprit.

Corresponding Homogeneous: $y'' + y = 0$. F.S. = {sin(x), cos(x)}.

If $y_p = v_1\cos(x) + v_2\sin(x)$, then v_1' and v_2' are solutions to the following system:

$$\begin{cases} \cos(x)\,v_1' + \sin(x)\,v_2' = 0 \\ -\sin(x)\,v_1' + \cos(x)\,v_2' = \sec^3(x) \end{cases}$$

Solving the system yields $v_1' = -\sec^2(x)\tan(x)$ and $v_2' = \sec^2(x)$. Thus, by integrating, we obtain

$$v_1 = -\frac{\tan^2(x)}{2} + c \ \ or \ \ v_1 = -\frac{\sec^2(x)}{2} + c \ \ and \ \ v_2 = \tan(x) + d.$$

Thus, a particular integral of the ODE above is

$$y_p = v_1\cos(x) + v_2\sin(x) = -\frac{\tan^2(x)\cos(x)}{2} + \tan(x)\sin(x) = \frac{\tan(x)\sin(x)}{2}$$

or

$$y_p = v_1\cos(x) + v_2\sin(x) = -\frac{\sec^2(x)\cos(x)}{2} + \tan(x)\sin(x) = \frac{\sec(x)}{2} - \cos(x)$$

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4. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' - 6y' + 10y = x^2e^{-x} + \cos(x)e^{3x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

First, the corresponding homogeneous equation is

$$y'' - 6y' + 10y = 0.$$

which has an auxiliary equation given by $0 = m^2 - 6m + 10$.
Thus, $m = 3 + i$ or $m = 3 - i$, and a fundamental set of solutions for the corresponding homogeneous equation is { exp(3x)cos(x), exp(3x)sin(x) }. Taking this into account, we may now write

$$y_p(x) = Ax^2e^{-x} + Bxe^{-x} + Ce^{-x} + Dx\cos(x)e^{3x} + Ex\sin(x)e^{3x}$$

or something equivalent.

Brief Bonus Answers:

(a) A fourth order constant coefficient homogeneous linear ODE with $f(x) = x^2$ and $g(x) = \exp(2x)$ as solutions:

$$y'''' - 2y''' = 0$$

(b) A second order homogeneous linear ODE with $f(x) = x^2$ and $g(x) = \exp(2x)$ as solutions:

$$(x^2-x)y'' + (1-2x^2)y' + (4x-2)y = 0\ .$$

5. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m^3(m-2)(m-\pi i)^2(m+\pi i)^2 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**]
(b) (5 pt.) What is the order of the differential equation?

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{2x}$$
$$+ c_5 \sin(\pi x) + c_6 \cos(\pi x)$$
$$+ c_7 x \sin(\pi x) + c_8 x \cos(\pi x)$$

The order of the differential equation is 8.

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6. (10 pts.) The nonzero function $f(x) = x$ is a solution to the homogeneous linear O.D.E.

$$(*) \qquad (x^2+1)y'' - 2xy' + 2y = 0.$$

(a) Reduction of order with this solution involves making the substitution

$$y = xv$$

into equation (*) and then letting $w = v'$. Do this substitution and obtain the first order linear homogeneous equation that w must satisfy.
(b) Finally, obtain an integrating factor, μ, for the first order linear ODE that w satisfies and then stop. ***Do not attempt to actually find v.***

(a) If we have

$$y = xv ,$$

then

$$y' = xv' + v , \text{ and } \quad y'' = xv'' + 2v' .$$

Substituting y into (*), and then replacing v' using w implies that w must be a solution to

$$w' + \frac{2}{x(x^2+1)} w = 0$$

after one cleans up the algebra a little.

(b) An integrating factor for the homogeneous linear equation that w satisfies is

$$\mu = \exp\left(\int \frac{2\,dx}{x(x^2+1)}\right) = \frac{x^2}{x^2+1}$$

since

$$\int \frac{2\,dx}{x(x^2+1)} = \int \frac{2}{x} - \frac{2x}{x^2+1}\,dx = \ln\left(\frac{x^2}{x^2+1}\right) + C .$$

Of course, a partial-fraction decomposition somewhere along the way helps.

7. (15 pts.) A body of mass 100 g is dropped from rest toward the earth from a height of 1000 m. As it falls, air resistance acts on it and the resistance, in newtons, is proportional to the velocity v, in meters per second. Suppose the limiting velocity is 245 m/sec.

(a) Using k to denote the unknown constant of proportionality, obtain an initial-value problem satisfied by the velocity, v, of the body.

(b) Solve the initial-value problem you gave in answering (a).

(c) Finally, using the information concerning the limiting velocity, determine k, and thus complete the determination of the velocity of the mass as a function of time.

Solution: (a) Use the drop point as the origin for our coordinate system and down as the positive direction. Then using Newton's Second Law and the mks system of units, the initial displacement is $x(0) = 0$, the initial velocity is $v(0) = 0$, and the velocity function $v(t)$ satisfies the ordinary differential equation

$$\frac{1}{10}\frac{dv}{dt} = 0.98 - kv \ ,$$

where the first term on the right side is the weight of the mass in newtons and the second term on the right is the air resistance. Note that we now have in hand the IVP that the velocity function must satisfy.

(b) Although the ODE above obviously is separable and can be solved as such, a minimal amount of algebra turns it into a linear constant coefficient equation with a UC driving function, thus:

$$(*) \qquad \frac{dv}{dt} + 10kv = 9.8 \ .$$

At this stage we can actually solve this without integrating. At any rate, solving the ODE and applying the initial condition $v(0) = 0$, results in us obtaining the velocity function:

$$v(t) = \frac{9.8}{10k} - \frac{(9.8)e^{-10kt}}{10k} \ .$$

(c) Since the terminal velocity is 245 m/sec,

$$245 = \lim_{t \to \infty} v(t) = \frac{9.8}{10k} \ .$$

It follows that the velocity function is

$$v(t) = 245 - 245e^{-t/25}$$

since $10k = 9.8/245 = 1/25$.

Silly 10 Point Bonus: Let $f(x) = x^2$ and $g(x) = \exp(2x)$. (a) It is trivial to obtain a 4th order homogeneous linear constant coefficient ordinary differential equation with f and g as solutions. Do so.
(b) It's only slightly messier to obtain a 2nd order homogeneous linear ordinary differential equation with $\{ f , g \}$ as a fundamental set of solutions. Do so. [Say where your work is, for it won't fit here.]
Brief answers are on the bottom of Page 2 of 4.