General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (30 pts.) Obtain the general solution to each of the following linear homogeneous constant coefficient equations.

(a) $y''(x) - 2y'(x) + 5y(x) = 0$

(b) $y''(x) - 2y'(x) - 8y(x) = 0$

(c) $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} = 0$

2. (10 pts.) Find the unique solution to the initial value problem

$$y'' - 2y' - 3y = -10\sin(x) \;;\quad y(0) = 2,\;\; y'(0) = 4,$$

given that a fundamental set of solutions to the corresponding homogeneous equation is { e^{3x}, e^{-x} } and a particular integral to the original ODE is

$$y_p(x) = 2\sin(x) - \cos(x).$$

Hint: Save time. Use the stuff served on the platter with the cherry on top.

3. (15 pts.) Find a particular integral, y_p, of the differential equation

$$y'' + y = \sec^3(x).$$

4. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' - 6y' + 10y = x^2 e^{-x} + \cos(x)\, e^{3x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

5. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m^3(m-2)(m-\pi i)^2(m+\pi i)^2 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation.
[**WARNING: Be very careful. This will be graded Right or Wrong!!**]
(b) (5 pt.) What is the order of the differential equation?

6. (10 pts.) The nonzero function $f(x) = x$ is a solution to the homogeneous linear O.D.E.

$$(*) \qquad (x^2+1)y'' - 2xy' + 2y = 0.$$

(a) Reduction of order with this solution involves making the substitution

$$y = xv$$

into equation (*) and then letting $w = v'$. Do this substitution and obtain the first order linear homogeneous equation that w must satisfy.
(b) Finally, obtain an integrating factor, μ, for the first order linear ODE that w satisfies and then stop. ***Do not attempt to actually find v.***

7. (15 pts.) A body of mass 100 g is dropped from rest toward the earth from a height of 1000 m. As it falls, air resistance acts on it and the resistance, in newtons, is proportional to the velocity v, in meters per second. Suppose the limiting velocity is 245 m/sec.

(a) Using k to denote the unknown constant of proportionality, obtain an initial-value problem satisfied by the velocity, v, of the body. [Hints: (i) mks; (ii) Making the drop point the origin simplifies matters.]

(b) Solve the initial-value problem you gave in answering (a).

(c) Finally, using the information concerning the limiting velocity, determine k, and thus complete the determination of the velocity of the mass as a function of time. [Obtaining the displacement after this is easy, but don't spend the time attempting to do so.]

Silly 10 Point Bonus: Let $f(x) = x^2$ and $g(x) = \exp(2x)$. (a) It is trivial to obtain a 4th order homogeneous linear constant coefficient ordinary differential equation with f and g as solutions. Do so.
(b) It's only slightly messier to obtain a 2nd order homogeneous linear ordinary differential equation with $\{ f , g \}$ as a fundamental set of solutions. Do so. [Say where your work is, for it won't fit here.]