Read Me First: Communicate. Show all essential work very neatly and use correct notation when presenting your computations and arguments. Write using complete sentences. Show me the all magic on the page. Eschew obfuscation.

Test #:

1. (10 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^2+x)^2 y'' + xy' + (x+1)y = 0$$

An equivalent equation in standard form is

$$y'' + \frac{x}{(x(x+1))^2} y' + \frac{x+1}{(x(x+1))^2} y = 0.$$

From this, we can see easily that $x_0 = 0$ is a regular singular point of the equation, and $x_0 = -1$ is an irregular singular point. All other real numbers are ordinary points of the equation.

2. (15 pts.) (a) If $f(t)$ and $g(t)$ are piece-wise continuous functions defined for $t \geq 0$, what is the definition of the convolution of f with g, $(f*g)(t)$??

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\,dx$$

(b) Suppose $f(t) = 6e^{3t}$ and $g(t) = 2e^{t}$.

Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$.

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\,dx = \int_0^t 6e^{3x}\cdot 2e^{t-x}\,dx$$
$$= 6e^{t}\int_0^t 2e^{2x}\,dx = 6e^{t}\left[(e^{2x})\big|_0^t\right] = \ldots = 6e^{3t} - 6e^{t}$$

(c) Suppose $h(t) = (f*g)(t),\ where\ f(t) = t^2\ and\ g(t) = \sin(2t)$.

Using the table, compute the Laplace transform of h.

$$\mathfrak{L}\{h(t)\}(s) = \mathfrak{L}\{(f*g)(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\,\mathfrak{L}\{g(t)\}(s)$$
$$= \mathfrak{L}\{t^2\}(s)\cdot\mathfrak{L}\{\sin(2t)\}(s)$$
$$= \left[\frac{2}{s^3}\right]\cdot\left[\frac{2}{s^2+4}\right]$$

Silly Bonus Work:

$$I = \int_0^\infty\left[\int_0^t x^2\sin(2(t-x))\,e^{-2t}\,dx\right]dt$$
$$= \int_0^\infty\left[\int_0^t x^2\sin(2(t-x))\,dx\right]e^{-2t}\,dt$$
$$= \mathfrak{L}\{(t^2*\sin(2t))\}(2) = \left[\frac{2}{2^3}\right]\cdot\left[\frac{2}{2^2+4}\right] = \frac{1}{16}$$

3. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - y' + x^2 y = 0.$$

Very neatly obtain the recurrence formula(s) needed to determine the coefficients of $y(x)$. DO NOT SPEND TIME ATTEMPTING TO GET THE NUMERICAL VALUES OF THE COEFFICIENTS.

First,

$$\begin{aligned} 0 &= x^2 y - y' + y'' \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\ &= \sum_{n=2}^{\infty} c_{n-2} x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \\ &= (2c_2 - c_1)x^0 + (6c_3 - 2c_2)x^1 + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} + c_{n-2}]x^n \end{aligned}$$

for all x near zero. From this you can deduce that we have

$$c_2 = \frac{c_1}{2},$$

$$c_3 = \frac{c_2}{3}, \text{ and}$$

$$c_{n+2} = \frac{(n+1)c_{n+1} - c_{n-2}}{(n+2)(n+1)} \quad \text{for } n \ge 2.$$

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4. (10 pts.) (a) Suppose that $f(t)$ is defined for $t > 0$. What is the definition of the Laplace transform of f in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt = \lim_{R\to\infty} \int_0^R f(t)e^{-st}\,dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0, & \text{if } 0 < t < 1 \\ 2, & \text{if } 1 < t. \end{cases}$$

$$\mathfrak{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^1 0e^{-st}\,dt + \int_1^R 2e^{-st}\,dt\right]$$

$$= \lim_{R\to\infty}\left[\frac{2e^{-s}}{s} - \frac{2e^{-Rs}}{s}\right] = \frac{2e^{-s}}{s} \quad \text{provided } s > 0.$$

Note: You may, of course, check your "answer" using #15 in the table.

5. (10 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' + 3xy' + (x^2-8)y = 0$$

(a) Obtain the indicial equation for the ODE at $x_0 = 0$ and its two roots.
(b) Then use all the information available and Theorem 6.3 to say what the two non-trivial linearly independent solutions given by the theorem look like without attempting to obtain the coefficients of the power series.
// To determine r, you need the indicial equation at $x_0 = 0$ and its roots. Now the indicial equation is $r(r-1) + p_0 r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{3x}{x^2}\right] = 3 \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{x^2-8}{x^2}\right] = -8.$$

Thus, the indicial equation is $r^2 + 2r - 8 = 0$, with two roots $r_1 = 2$ and $r_2 = -4$. Consequently the two linearly independent solutions provided by Theorem 6.3 look like the following:

$$y_1(x) = |x|^2\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2(x) = |x|^{-4}\sum_{n=0}^{\infty} d_n x^n + C y_1(x)\ln|x|$$

6. (10 pts.) Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{5s+17}{s^2+4s+13} \overset{(Work)}{=} \dfrac{5s+17}{(s+2)^2+9} = \dfrac{5(s+2)+7}{(s+2)^2+3^2}$

$$\mathcal{L}^{-1}\{F(s)\}(t) = 5e^{-2t}\cos(3t) + \frac{7}{3}e^{-2t}\sin(3t)$$

(b) $F(s) = \dfrac{5}{s^2+4s+4} \overset{(Work)}{=} \dfrac{5}{(s+2)^2}$

$$\mathcal{L}^{-1}\{F(s)\}(t) = 5te^{-2t}$$

'Tis just the usual magic of multiplication by '1' in the correct form or the addition of '0' suitably transmogrified with linearity tossed into the mix.

7. (5 pts.) Circle the letter corresponding to the correct response: If $F(s) = \mathcal{L}\{t\sin(bt)e^{at}\}(s)$,then $F(s) =$

(a) $\left[\dfrac{1}{s^2}\right]\cdot\left[\dfrac{b}{s^2+b^2}\right]\cdot\left[\dfrac{1}{s-a}\right]$ (b) $\dfrac{d}{ds}\left[\dfrac{b}{(s-a)^2+b^2}\right]$

(c) $\dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$ (d) $\left[\dfrac{2bs}{(s^2+b^2)^2}\right]\cdot\left[\dfrac{1}{s-a}\right]$

(e) $\left[\dfrac{b}{s^2+b^2}\right]\cdot\left[\dfrac{1}{(s-a)^2}\right]$ (f) None of (a) through (e).

Obviously (a), (d), and (e) are utter nonsense since the transform of a product is NOT the product of the transforms. (b) is a near miss since the sign is wrong. It turns out that (c) provides the correct answer.

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8. (15 pts.) Transform the given initial value problem into an algebraic equation in $\mathfrak{L}\{y\}$ and solve for $\mathfrak{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$y''(t) - 6y'(t) + 5y(t) = 0 \; ; \quad y(0) = 3 \, , \quad y'(0) = 7 \; .$$

Applying the Laplace transform operator to BOTH SIDES OF THE ODE, using the two initial conditions, and then solving for the transform of y should reveal that

$$\mathfrak{L}\{y(t)\}(s) = \left[\frac{1}{s^2-6s+5}\right]\cdot[3s - 11]$$

[A common error: Failure to parenthesize the first and second derivative's transform correctly.]

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9. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$-c_2 = 0, \quad c_0 + 3c_1 - 6c_3 = 0, \text{ and}$$

$$c_n = \frac{(n-2)\,n\,c_{n-2} + c_{n-3}}{n(n-1)} \quad \textit{for all } n \geq 4.$$

Determine the exact numerical value of the coefficients c_0, c_1, c_2, c_3, and c_4 for the particular solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$c_0 = y(0) = 1 \qquad\qquad c_1 = y'(0) = 0$$

$$c_2 = 0 \qquad\qquad c_3 = \frac{c_0 + 3c_1}{6} = \frac{1}{6}$$

$$c_4 = \frac{8c_2 + c_1}{(4)(3)} = \frac{0}{12} = 0$$

Silly 10 Point Bonus: Would you believe that you have already done the hard work in evaluating the following improper integral??

$$I = \int_0^{\infty}\left[\int_0^{t} x^2 \sin(2(t-x))\, e^{-2t}\, dx\right] dt$$

What is the numerical value of I? Say where your work is, for it might not fit here.
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Toidi's work is on the bottom of Page 1 of 4, for the obvious reason. Well --- maybe not.