Student Number: 0000000 Exam Number: 00

Read Me First:

Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Good Luck! [Total Points: 160]

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1. (80 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(a) $(2xy - 3) + (x^2 + 4y)\frac{dy}{dx} = 0$; $y(1) = 2$ The differential equation in (a) is equivalent to $(2xy - 3)dx + (x^2 + 4y)dy = 0$.

Since $\frac{\partial}{\partial y}(2xy - 3) = 2x = \frac{\partial}{\partial x}(x^2 + 4y)$, this equation is exact. A one-parameter family of implicit solutions is given by $x^2y - 3x + 2y^2 = C$. Using the initial condition to determine C results in the implicit solution $x^2y - 3x + 2y^2 = 7$.

(b) $y'' - 2y' + y = \frac{e^x}{x^3}$ First, the corresponding homogeneous equation is

$$y'' - 2y' + y = 0 \;,$$

which has an auxiliary equation given by $0 = m^2 - 2m + 1$. Thus, $m = 1$ with multiplicity 2, and a fundamental set of solutions for the corresponding homogeneous equation is

$$F.S. = \{e^x, xe^x\} \;.$$

Obviously the driving function here is NOT a UC function. Thus, we shall use variation of parameters to nab a particular integral for the ODE. [Note: A particular integral can be obtained by other means, though.] If

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x \;,$$

then v_1' and v_2' are solutions to the following system:

$$\begin{cases} e^x v_1' + xe^x v_2' = 0 \\ e^x v_1' + [xe^x + e^x]v_2' = x^{-3}e^x \end{cases}$$

Solving the system yields $v_1' = -x^{-2}$ and $v_2' = x^{-3}$. Thus, by integrating, we obtain $v_1(x) = x^{-1} + c$ *and* $v_2(x) = -\frac{1}{2}x^{-2} + d$. Thus, a particular integral of the ODE above is

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x = x^{-1}e^x - \tfrac{1}{2}x^{-2}xe^x = \tfrac{1}{2}x^{-1}e^x.$$

Consequently, the general solution is

$$y(x) = c_1e^x + c_2xe^x + \tfrac{1}{2}x^{-1}e^x.$$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(c) $x\dfrac{dy}{dx} - 2y = 2x^4$; $y(2) = 8$

The ODE of problem (c) is linear as written, but it is not in standard form with the coefficient on the derivative being one. Here is the standard form version:

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^3$$

It has an obvious integrating factor of $\mu = x^{-2}$ for $x > 0$. Multiplying the ODE by μ, integrating, and then dealing with the initial condition allows you to produce $x^{-2}y = x^2 - 2$, an implicit solution. An obvious explicit solution is $y(x) = x^4 - 2x^2$, actually defined for all x.

(d) $y'' + y = 4x^2$

First, the corresponding homogeneous equation is

$$y'' + y = 0 \ ,$$

which has an auxiliary equation given by $0 = m^2 + 1$. Thus, $m = \pm i$, and a fundamental set of solutions for the corresponding homogeneous equation is

$$F.S. = \{\cos(x), \sin(x)\} \ .$$

Obviously the driving function here is a UC function. Thus, we may use the UC machinery to nab a particular integral for the ODE. A UC set is given by

$$U.C. = \{x^2, x, 1\}$$

If

$$y_p(x) = Ax^2 + Bx + C \ ,$$

and this is a solution to the ODE of (d), then

$$\begin{cases} A = 4 \\ B = 0 \\ 2A + C = 0 \end{cases}$$

Solving the system yields $A = 4$, $B = 0$, and $C = -8$. Thus, a particular integral of the ODE above is

$$y_p(x) = 4x^2 - 8 \ .$$

Consequently, the general solution is

$$y(x) = c_1\cos(x) + c_2\sin(x) + 4x^2 - 8 \ .$$

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2. (8 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathcal{L}\{y(t)\}(s) = \frac{1 + e^{-(\pi/2)s}}{s^2 + 4}$$

Write the solution to the IVP in piecewise-defined form.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{e^{-(\frac{\pi}{2})t}}{s^2+4}\right\}(t)$$

$$= \frac{1}{2}\sin(2t) + u_{\frac{\pi}{2}}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t - \tfrac{\pi}{2})$$

$$= \frac{1}{2}\sin(2t) + \frac{1}{2}u_{\frac{\pi}{2}}(t)\sin(2(t - \tfrac{\pi}{2}))$$

$$= \begin{cases} \frac{1}{2}\sin(2t), & for\ 0 \le t < \frac{\pi}{2} \\ 0, & for\ \quad t > \frac{\pi}{2} \end{cases}$$

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3. (12 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows. (4 pts./part)

(a) $h(t) = 4t^2\sin^2(t) = 4t^2\left[\frac{1 - \cos(2t)}{2}\right] = 2t^2 - 2t^2\cos(2t)$

$$\mathcal{L}\{h(t)\}(s) = \frac{4}{s^3} - 2\mathcal{L}\{t(t\cos(2t))\}(s) = \frac{4}{s^3} + 2\frac{d}{ds}[\mathcal{L}\{t\cos(2t)\}(s)]$$

$$= \frac{4}{s^3} + 2\frac{d}{ds}\left[\frac{s^2 - 4}{(s^2+4)^2}\right] = \frac{4}{s^3} + 2\left[\frac{24s - 2s^3}{(s^2+4)^3}\right]$$

(b)

$$f(t) = \begin{cases} 1, & if\ 0 < t < 2 \\ 2, & if\ 2 < t < 4 \\ 3, & if\ 4 < t < 6 \\ 0, & if\ 6 < t. \end{cases} = 1 + u_2(t) + u_4(t) - 3u_6(t)$$

$$\mathcal{L}\{f(t)\}(s) = \frac{1}{s} + \frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} - \frac{3e^{-6s}}{s}$$

(c)

$$g(t) = \begin{cases} 2t, & if\ 0 < t < 5 \\ 10, & if\ 5 < t. \end{cases} = 2t + (10 - 2t)u_5(t)$$

$$\mathcal{L}\{g(t)\}(s) = \frac{2}{s^2} + \mathcal{L}\{(10 - 2t)u_5(t)\}(s)$$

$$= \frac{2}{s^2} + \mathcal{L}\{f(t-5)u_5(t)\}(s),\ where\ f(t-5) = 10 - 2t$$

$$= \frac{2}{s^2} + e^{-5s}\mathcal{L}\{f(t)\}(s),\ where\ f(t) = f((t+5)-5) = -2t$$

$$= \frac{2}{s^2} + e^{-5s}\mathcal{L}\{-2t\}(s) = \frac{2}{s^2} - \frac{2e^{-5s}}{s^2}$$

4. (10 pts.) Obtain the recurrence formula(s) satisfied by the coefficients of the power series solution y at $x_0 = 0$, an ordinary point of the homogeneous ODE,

$$y'' - y' + 2xy = 0.$$

First,

$$\begin{aligned} 0 &= 2xy - y' + y'' \\ &= 2x\sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\ &= \sum_{n=1}^{\infty} 2c_{n-1} x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \\ &= (2c_2 - c_1)x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} + 2c_{n-1}]x^n. \end{aligned}$$

From this you can deduce that $c_2 = c_1/2$, and that for $n \geq 1$, we have

$$c_{n+2} = \frac{(n+1)c_{n+1} - 2c_{n-1}}{(n+2)(n+1)}.$$

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5. (10 pts.) (a) (4 pts.) Obtain the differential equation satisfied by the family of curves defined by the equation (*) below.

(b) (3 pts.) Next, write down the differential equation that the orthogonal trajectories to the family of curves defined by (*) satisfy.

(c) (3 pts.) Finally, solve the differential equation of part (b) to obtain the equation(s) defining the orthogonal trajectories.

$$(*) \qquad y = e^{cx}$$

(a) $\dfrac{dy}{dx} = ce^{cx} = \dfrac{\ln(y)}{x}e^{\ln(y)} = \dfrac{y\ln(y)}{x}$

(b) $\dfrac{dy}{dx} = -\dfrac{x}{y\ln(y)}$, *which plainly is separable.*

(c) The differential equation in part (b) has no constant solutions, when y is considered a function of x, and separating variables yields

$$x\,dx + y\ln(y)\,dy = 0 .$$

Integrating, and clearing the common denominator provides us with

$$2x^2 + 2y^2\ln(y) - y^2 = K ,$$

an equation for the orthogonal trajectories.

6. (10 pts.) The equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

has a regular singular point at $x_0 = 0$. Substitution of

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \text{ for } x > 0,$$

into the ODE and a half a page of algebra yields

$$(r^2-1)c_0x^r + ((1+r)^2-1)c_1x^{r+1} + \sum_{n=2}^{\infty}\left([(n+r)^2 - 1]c_n + c_{n-2}\right)x^{n+r} = 0.$$

Using this information, (a) write the form of the two linearly independent solutions to the ODE given by Theorem 6.3 without obtaining the numerical values of the coefficients of the series involved, and then (b) do obtain the the numerical values of the coefficients c_1, ... , c_5 to the first solution, $y_1(x)$, given by Theorem 6.3 when $c_0 = 1$. [Keep the parts separate.]

(a) Plainly, the indicial equation is $r^2 - 1 = 0$, with two roots $r_1 = 1$ and $r_2 = -1$. Consequently the two linearly independent solutions provided by Theorem 6.3 look like the following:

$$y_1(x) = |x|^1 \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2(x) = |x|^{-1}\sum_{n=0}^{\infty} d_n x^n + C y_1(x)\ln|x| .$$

(b) When $r = r_1 = 1$, $(1+r)^2 - 1 \neq 0$. Thus, we must have $c_1 = 0$, and

$$c_n = -\frac{c_{n-2}}{n(n+2)} \quad \text{for } n \geq 2.$$

Using this recurrence and the given information, it follows that

$$c_0 = 1, c_1 = c_3 = c_5 = 0, c_2 = -\frac{1}{(2)(4)}, \text{and } c_4 = \frac{1}{(2)(4^2)(6)}.$$

7. (10 pts.) Solve the following second order initial-value problem using only the Laplace transform machine.

$$y''(t) + y(t) = \cos(t) \quad ; \; y(0) = 3, \; y'(0) = 2.$$

By taking the Laplace Transform of both sides of the differential equation, and using the initial conditions, we have

$$\mathfrak{L}\{y''(t)\}(s) + \mathfrak{L}\{y(t)\}(s) = \mathfrak{L}\{\cos(t)\}$$

$$\Rightarrow \quad s^2\mathfrak{L}\{y(t)\}(s) - sy(0) - y'(0) + \mathfrak{L}\{y(t)\}(s) = \frac{s}{s^2+1}$$

$$\Rightarrow \quad (s^2+1)\mathfrak{L}\{y(t)\}(s) = 3s + 2 + \frac{s}{s^2+1}$$

$$\Rightarrow \quad \mathfrak{L}\{y(t)\}(s) = \frac{3s}{s^2+1} + \frac{2}{s^2+1} + \frac{s}{(s^2+1)^2}.$$

By taking inverse transforms now, we quickly obtain

$$y(t) = 3\cos(t) + 2\sin(t) + \frac{1}{2}t\sin(t).$$

8. (10 pts.) (a) Given $f(x) = x$ is a nonzero solution to

$$(x^2-1)y'' - 2xy' + 2y = 0 \ ,$$

obtain a second, linearly independent solution by reduction of order.
(b) Use the Wronskian to prove the two solutions are linearly independent.
(a) Set $y = vx$. Substituting y into the ODE above and simplifying the algebra results in

$$(x^3-x)v'' - 2v' = 0.$$

Then, setting $w = v'$, and a little additional algebra provides us with the linear, 1st order, homogeneous equation

$$w' - \left(\frac{2}{x^3-x}\right)w = 0.$$

An integrating factor for this ODE is $\mu = x^2/(x^2 - 1)$. Solving the ODE, an additional integration, and a suitable choice of constants allows us to obtain a second solution

$$g(x) = x^2 + 1.$$

$$W(f,g)(x) = \begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix} = x^2 - 1 \neq 0 \ \ when\ x \neq \pm 1.$$

Hence, f and g are linearly independent functions.

9. (10 pts.) An 8-lb weight is attached to the lower end of a coil spring suspended from the ceiling and comes to rest in its equilibrium position, thereby stretching the spring 0.4 ft. The weight is then pulled down 6 inches below its equilibrium position and released at t = 0. The resistance of the medium in pounds is numerically equal to 2x′, where x′ is the instantaneous velocity in feet per second.

(a) Set up the differential equation for the motion and list the initial conditions.
(b) Solve the initial-value problem set up in part (a) to determine the displacement of the weight as a function of time.

(a) Let $x(t)$ denote the displacement, in feet, from the equilibrium position at time t, in seconds. Since the mass, $m = 8/32 = 1/4$ slug, and the spring constant is $k = 20$ pounds per foot, an initial-value problem describing the motion of the mass is

$$\begin{cases} \frac{1}{4}x''(t) + 2x'(t) + 20x(t) = 0 \\ with\ \ x(0) = \frac{1}{2} \ and\ x'(0) = 0. \end{cases}$$

(b) The ODE is equivalent to $x'' + 8x' + 80x = 0$ with

$$x(t) = c_1 e^{-4t}\cos(8t) + c_2 e^{-4t}\sin(8t)$$

as the general solution. Using the initial conditions to build a linear system and solving the system provides us with the solution to the IVP,

$$x(t) = \frac{1}{2}e^{-4t}\cos(8t) + \frac{1}{4}e^{-4t}\sin(8t) \ .$$

Bonkers 10 Point Bonus: Obtain a condition that implies that $\mu(x)$ will be an integrating factor of the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

and show how to compute μ when that sufficient condition is true. Say where your work is for it won't fit here.