NAME:

Student Number: Exam Number:

Read Me First:

Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Remember that what is illegible or incomprehensible is worthless. Communicate. Good Luck! [Total Points: 160]

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1. (80 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $(2xy - 3) + (x^2 + 4y)\dfrac{dy}{dx} = 0$; $y(1) = 2$

(b) $y'' - 2y' + y = \dfrac{e^x}{x^3}$

1. (Continued) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(c) $x\frac{dy}{dx} - 2y = 2x^4$; $y(2) = 8$

(d) $y'' + y = 4x^2$

2. (8 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathfrak{L}\{y(t)\}(s) = \frac{1 + e^{-(\pi/2)s}}{s^2 + 4}$$

Write the solution to the IVP in piecewise-defined form.

$y(t)$ =

3. (12 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows. (4 pts./part)

(a) $h(t) = 4t^2 \sin^2(t)$ =

$\mathfrak{L}\{h(t)\}(s)$ =

(b)

$$f(t) = \begin{cases} 1, & if \;\; 0 < t < 2 \\ 2, & if \;\; 2 < t < 4 \\ 3, & if \;\; 4 < t < 6 \\ 0, & if \;\; 6 < t. \end{cases} =$$

$\mathfrak{L}\{f(t)\}(s)$ =

(c)

$$g(t) = \begin{cases} 2t, & if \;\; 0 < t < 5 \\ 10, & if \;\; 5 < t. \end{cases} =$$

$\mathfrak{L}\{g(t)\}(s)$ =

4. (10 pts.) Obtain the recurrence formula(s) satisfied by the coefficients of the power series solution y at $x_0 = 0$, an ordinary point of the homogeneous ODE,

$$y'' - y' + 2xy = 0.$$

5. (10 pts.) (a) (4 pts.) Obtain the differential equation satisfied by the family of curves defined by the equation (*) below.

(b) (3 pts.) Next, write down the differential equation that the orthogonal trajectories to the family of curves defined by (*) satisfy.

(c) (3 pts.) Finally, solve the differential equation of part (b) to obtain the equation(s) defining the orthogonal trajectories.

$$(*) \qquad y = e^{cx}$$

6. (10 pts.) The equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

has a regular singular point at $x_0 = 0$. Substitution of

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}, \text{ for } x > 0,$$

into the ODE and a half a page of algebra yields

$$(r^2-1)c_0x^r + ((1+r)^2-1)c_1x^{r+1} + \sum_{n=2}^{\infty} \left([(n+r)^2 - 1]c_n + c_{n-2}\right)x^{n+r} = 0.$$

Using this information, (a) write the form of the two linearly independent solutions to the ODE given by Theorem 6.3 without obtaining the numerical values of the coefficients of the series involved, and then (b) do obtain the the numerical values of the coefficients c_1, ... , c_5 to the first solution, $y_1(x)$, given by Theorem 6.3 when $c_0 = 1$. [Keep the parts separate.]

7. (10 pts.) Solve the following second order initial-value problem using only the Laplace transform machine.

$$y''(t) + y(t) = \cos(t) \quad ; \; y(0) = 3, \; y'(0) = 2.$$

8. (10 pts.) (a) Given $f(x) = x$ is a nonzero solution to

$$(x^2-1)y'' - 2xy' + 2y = 0 \ ,$$

obtain a second, linearly independent solution by reduction of order.
(b) Use the Wronskian to prove the two solutions are linearly independent.

9. (10 pts.) An 8-lb weight is attached to the lower end of a coil spring suspended from the ceiling and comes to rest in its equilibrium position, thereby stretching the spring 0.4 ft. The weight is then pulled down 6 inches below its equilibrium position and released at t = 0. The resistance of the medium in pounds is numerically equal to 2x′, where x′ is the instantaneous velocity in feet per second.

(a) Set up the differential equation for the motion and list the initial conditions.
(b) Solve the initial-value problem set up in part (a) to determine the displacement of the weight as a function of time.

Bonkers 10 Point Bonus: Obtain a condition that implies that $\mu(x)$ will be an integrating factor of the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

and show how to compute μ when that sufficient condition is true. Say where your work is for it won’t fit here.