General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Write complete sentences, and use notation correctly. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

1. (80 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [20 points/part]

(a) $\frac{dy}{dx} = \tan^2(x)\sec(y)$; $y(0) = \frac{\pi}{6}$. *Separable* as written.

Since sec(y) = 0 has no solutions, there are no immediate constant solutions. By separating variables, we have

$$\tan^2(x)\,dx - \cos(y)\,dy = 0 .$$

Integrating and applying the initial condition leads to the implicit solution

$$\sin(y) = \tan(x) - x + \frac{1}{2} .$$

An explicit solution that is valid near zero is

$$y(x) = \sin^{-1}\left(\tan(x) - x + \frac{1}{2}\right) .$$

(b) $x\frac{dy}{dx} + y = (xy)^{3/2}$ *for* $x > 0$. *Bernoulli*, obviously.

The substitution $v = y^{-1/2}$ allows us to reduce (b) to a linear ODE in *v* and *x*, namely

$$\frac{dv}{dx} - \frac{1}{2x}v = -\frac{1}{2}x^{1/2}. .$$

Solving this and substituting back results in a 1-parameter family of implicit solutions:

$$y^{-1/2} = cx^{1/2} - \frac{1}{2}x^{3/2} .$$

Getting an explicit solution now is cheap thrills.

Bonus Noise: The substitution *v* = *y′* converts the ODE

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

into the first order linear equation

$$(*) \quad \frac{dv}{dx} + v = x.$$

Using the integrating factor $\mu = e^x$ leads to the explicit solution to (*)

$$v(x) = x - 1 + ce^{-x} .$$

Finally, $y(x) = \int v(x)\,dx = \int x - 1 + ce^{-x}\,dx = \frac{x^2}{2} - x - ce^{-x} + d$.

(c) $\frac{dr}{d\theta} + 2\tan(\theta)r = 4\cos^3(\theta)$; $r(0) = 4$ *Linear* as written.

Near θ = 0, an integrating factor is easy to come by:

$$\mu(\theta) = e^{\int 2\tan(\theta)d\theta} = e^{2\ln|\sec(\theta)|} = \sec^2(\theta)$$

for θ ε (-π/2,π/2). Multiplying both sides of the DE by μ results in the following derivative equation:

$$\frac{d}{d\theta}(\sec^2(\theta)r(\theta)) = 4\cos(\theta).$$

By integrating, we have

$$\sec^2(\theta)r(\theta) = \int \frac{d}{d\theta}(\sec^2(\theta)r(\theta))\ d\theta = \int 4\cos(\theta)\ d\theta$$
$$= 4\sin(\theta) + C.$$

Applying the initial condition, it follows that C = 4. Hence, an explicit solution near θ = 0 is given by

$$r(\theta) = (4\sin(\theta) + 4)\cos^2(\theta).$$

(d) $(x+2y) + (2x+y)\frac{dy}{dx} = 0$

After minimal algebra, this may be seen as either *exact* with an easy solution, or *homogeneous*, of degree 1, with a messier solution.

As Exact: The usual prestidigitation leads to

$$F(x,y) = \frac{x^2}{2} + 2xy + \frac{y^2}{2} + c_0.$$

A 1-parameter family of solutions: $\frac{x^2}{2} + 2xy + \frac{y^2}{2} = C$, *or equivalent*.

As Homogeneous: The degree of homogeneity is 1. Thus, write the equation in the form of *dy/dx* = *g*(*y/x*) by doing suitable algebra carefully. After setting *y* = *vx*, substituting, and doing a bit more algebra, you will end up looking at the separable equation

$$(v^2 + 4v + 1)dx + x(v + 2)dv = 0 .$$

Separating variables and integrating leads you to

$$\int \frac{1}{x}\ dx + \int \frac{v + 2}{v^2 + 4v + 1}\ dv = C .$$

After doing that and completing the integration, you'll obtain

$$\ln|x| + \frac{1}{2}\ln|v^2 + 4v + 1| = C$$

or an equivalent beast. You may then finish this by replacing *v* above with *y/x*. Additional algebra will lead to the same 1-parameter family of implicit solutions as the *exact* route.

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2. (6 pts.) What conditions on the coefficients of the following homogeneous equation are sufficient for the equation to be exact?

$$(Ax^2 + Bxy + Cy^2)dx + (Dx^2 + Exy + Fy^2)dy = 0$$

Do not attempt to solve the DE.

Since

$$\frac{\partial}{\partial y}(Ax^2 + Bxy + Cy^2) = Bx + 2Cy \text{ and } \frac{\partial}{\partial x}(Dx^2 + Exy + Fy^2) = 2Dx + Ey\ ,$$

and polynomials of two variables have continuous second order partial derivatives on the whole of the xy-plane, the equation will be exact precisely when

$$B = 2D \text{ and } 2C = E.$$

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3. (6 pts.) Every solution to the differential equation $y'' - 16y = 0$ is of the form $y(x) = c_1 e^{4x} + c_2 e^{-4x}$. Which of these functions satisfies the initial conditions $y(0) = 4$ and $y'(0) = 8$??

The initial conditions lead to the system of equations

$$\begin{cases} 4 = c_1 + c_2 \\ 8 = 4c_1 - 4c_2 \end{cases} \text{ which is equivalent to } \begin{cases} c_1 = 3 \\ c_2 = 1 \end{cases}$$

The solution to the IVP is given by

$$y(x) = 3e^{4x} + e^{-4x}\ .$$

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4. (8 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain.**

$$(x - 2y + 2)dx + (4x - 8y - 6)dy = 0$$

The key to this puzzle is the solution to the linear system

$$\begin{cases} h - 2k + 2 = 0 \\ 4h - 8k - 6 = 0 \end{cases} \text{ which is equivalent to } \begin{cases} h - 2k = -2 \\ h - 2k = \dfrac{6}{4} \end{cases},$$

geometrically, a pair of parallel lines. Consequently, a suitable substitution is given by $z = x - 2y$. After a little routine algebra, the reduction results in the separable DE

$$(3z - 1)dx + (2z - 3)dz = 0$$

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10 Point Bonus: Although you will later learn a body of theory that will make the solution of the following following 2rd order equation routine, the truth is that you can actually obtain the general solution to the ODE with the knowledge you have now.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

Show how in detail. [Say where your work is below! There isn't room here.]

The is done briefly on the bottom of Page 1 of 3.