General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Eschew Obfuscation. Show me all the magic on the page.*

1. (30 pts.) Obtain the general solution to each of the following linear homogeneous constant coefficient equations.

(a) $y''(x) - 7y'(x) + 10y(x) = 0$

Auxiliary Equation: $0 = m^2 - 7m + 10 = (m-2)(m-5)$

Roots of A.E.: $m = 2$ *or* $m = 5$

General Solution: $y = c_1 e^{2x} + c_2 e^{5x}$

(b) $y''(x) - 4y'(x) + 5y(x) = 0$

Auxiliary Equation: $0 = m^2 - 5m + 5 = ((m-2)+i)((m-2)-i)$

Roots of A.E.: $m = 2+i$ *or* $m = 2-i$.

General Solution: $y = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x)$

(c) $\dfrac{d^5y}{dx^5} + 4\dfrac{d^3y}{dx^3} = 0$

Auxiliary Equation: $0 = m^5 + 4m^3 = m^3(m^2+4) = m^3(m+2i)(m-2i)$

Roots of A.E.: $m = 0$ *with multiplicity* 3, *or* $m = \pm 2i$

General Solution: $y = c_1 + c_2 x + c_3 x^2 + c_4\cos(2x) + c_5\sin(2x)$

2. (10 pts.) Find the unique solution to the initial value problem

$$y'' - y' = 2e^x \quad ; \quad y(0) = 2, \quad y'(0) = -2,$$

given that a fundamental set of solutions to the corresponding homogeneous equation is { 1, e^x } and a particular integral of the original ODE is

$$y_p(x) = 2xe^x.$$

Hint: Save time. Use the stuff served on the platter with the cherry on top.

The general solution to the ODE is

$$y(x) = c_1 + c_2 e^x + 2xe^x.$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = 6 - 4e^x + 2xe^x.$$

3. (15 pts.) Find a particular integral, y_p, of the differential equation

$$y'' + 4y = 4\,csc(2x).$$

Obviously the driving function here is NOT a UC function. Thus, we must use variation of parameters to nab the culprit.

Corresponding Homogeneous: $y'' + 4y = 0$. F.S. = {sin(2x), cos(2x)}.

If $y_p = v_1\cos(2x) + v_2\sin(2x)$, then v_1' and v_2' are solutions to the following system:

$$\begin{cases} \cos(2x)\,v_1' + \sin(2x)\,v_2' = 0 \\ -2\sin(2x)\,v_1' + 2\cos(2x)\,v_2' = 4\,csc(2x) \end{cases}$$

Solving the system yields $v_1' = -2$ and $v_2' = 2\cot(2x)$. Thus, by integrating, we obtain

$$v_1 = -2x + c \;\; and \;\; v_2 = \ln|\sin(2x)| + d.$$

Thus, a particular integral of the ODE above is

$$y_p = v_1\cos(x) + v_2\sin(x) = -2x\cos(x) + \ln|\sin(2x)|\sin(x) \;.$$

---

4. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' - 6y' + 10y = 4\sin(x) + 5xe^{3x} - 10\cos(x)e^{3x}.$$

First, the corresponding homogeneous equation is

$$y'' - 6y' + 10y = 0.$$

which has an auxiliary equation given by $0 = m^2 - 6m + 10$.
Thus, $m = 3 + i$ or $m = 3 - i$, and a fundamental set of solutions for the corresponding homogeneous equation is { exp(3x)cos(x), exp(3x)sin(x) }. Taking this into account, we may now write

$$y_p(x) = A\cos(x) + B\sin(x) + Cxe^{3x} + De^{3x} + Ex\cos(x)e^{3x} + Fx\sin(x)e^{3x}$$

Bonus Noise: If $f(x) = x^n$, then the first n derivatives are nonzero and given by

$$f^{(k)}(x) = \frac{(n!)}{(n-k)!}x^{n-k} \quad for\ k = 1, \ldots, n.$$

Higher order derivatives are zero. If f is a solution to

(*) $$a_m y^{(m)} + a_{m-1}y^{(m-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0$$

where $m > n$, then by direct substitution

(**) $$\sum_{k=0}^{n} a_k\left(\frac{n!}{(n-k)!}\right)x^{n-k} = 0.$$

The left side of (**) is a nonzero polynomial of degree at most n if there is some a_k nonzero for some k with $0 \le k \le n$. Such a polynomial has at most n roots. So in this case, we would not have a function identity on any interval of the line. It follows that $a_k = 0$ for every k with $0 \le k \le n$. Consequently, the fundamental set of solutions includes the following:

$$1, x, x^2, \ldots, x^{n-1}, x^n.$$

5. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$(m - 1)^4 (m - (\pi + i))^2 (m + (\pi - i))^2 = 0$$

Note: The plus sign in the factor $(m + (\pi - i))^2$ above is a typographical error.

(a) (5 pts.) Write down the general solution to the differential equation.
[**WARNING: Be very careful. This will be graded Right or Wrong!!**]
(b) (5 pt.) What is the order of the differential equation?

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x$$
$$+ c_5 \sin(x) e^{\pi x} + c_6 \cos(x) e^{\pi x}$$
$$+ c_7 x \sin(x) e^{\pi x} + c_8 x \cos(x) e^{\pi x}$$

The order of the differential equation is 8.

6. (15 pts.) (a) Obtain the differential equation satisfied by the family of curves defined by the equation (*) below.
(b) Next, write down the differential equation that the orthogonal trajectories to the family of curves defined by (*) satisfy.
(c) Finally, solve the differential equation of part (b) to obtain the equation(s) defining the orthogonal trajectories. [These, after all, are another family of curves.]

$$(*) \qquad y = x - 1 + ce^{-x} .$$

(a) Differentiating (*) with respect to x and then replacing *c* using (*) yields

$$\frac{dy}{dx} = 1 - ce^{-x}$$
$$= 1 - (y - x + 1)$$
$$= x - y,$$

a differential equation for the family of curves.
(b) An ODE for the orthogonal trajectories is now given by

$$\frac{dy}{dx} = -\frac{1}{x-y} = \frac{1}{y-x} ,$$

or equivalently,

$$(y-x)\frac{dy}{dx} = 1. \quad .$$

(c) This little equation is equivalent to one that is linear with x as the dependent variable and y the independent variable(!):

$$\frac{dx}{dy} + x = y.$$

This linear equation has $\mu = \exp(y)$ as an integrating factor. Using the standard recipe, a one-parameter family of solutions is given by

$$x = y - 1 + Ke^{-y} . \quad .$$

7. (10 pts.) The nonzero function $f(x) = \exp(x)$ is a solution to the homogeneous linear O.D.E.

(*) $$(x-1)y'' - xy' + y = 0$$

(a) Reduction of order with this solution involves making the substitution

$$y = ve^x$$

into equation (*) and then letting $w = v'$. Do this substitution and obtain in standard form the first order linear homogeneous equation that w must satisfy. (b) Finally, obtain an integrating factor, μ, for the first order linear ODE that w satisfies and then stop. ***Do not attempt to actually find v.***

(a) If we have

$$y = ve^x ,$$

then

$$y' = v'e^x + ve^x , \text{ and } \quad y'' = v''e^x + 2v'e^x + ve^x .$$

Substituting y into (*), and then replacing v' using w implies that w must be a solution to

$$w' + \left(\frac{x-2}{x-1}\right)w = 0$$

after one cleans up the algebra a little.

(b) An integrating factor for the homogeneous linear equation that w satisfies is

$$\mu = \exp\left(\int \frac{x-2}{x-1}dx\right) = \frac{e^x}{x-1}$$

since

$$\int \frac{x-2}{x-1}\, dx = \int 1 - \frac{1}{x-1}\, dx = x - \ln|x-1| + C .$$

Silly 10 Point Bonus: Suppose that the function

$$f(x) = x^n ,$$

where n is a positive integer, is a solution to the constant coefficient homogeneous linear equation

(*) $$a_m y^{(m)} + a_{m-1} y^{(m-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0$$

where $m > n$. What can you say about the coefficients of the ODE, and what can you say about its fundamental set of solutions?? Why???
[Say where your work is, for it won't fit here.]

Appropriate noise may be found on the bottom of Page 2 of 4.