**Read Me First**: Communicate. Show all essential work very neatly and use correct notation when presenting your computations and arguments. Write using complete sentences. Show me the all magic on the page. Eschew obfuscation.

Test #:

1. (15 pts.) (a) If $f(t)$ and $g(t)$ are piece-wise continuous functions defined for $t \geq 0$, what is the definition of the convolution of $f$ with $g$, $(f*g)(t)$??

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\;dx$$

(b) Suppose $f(t) = e^{3t}$ and $g(t) = 2e^{-5t}$ .

Using only the definition of the convolution as a definite integral, not some fancy transform shenanigans, compute $(f*g)(t)$.

$$(f*g)(t) = \int_0^t f(x)\,g(t-x)\;dx = \int_0^t e^{3x}2e^{-5(t-x)}\;dx$$
$$= 2e^{-5t}\int_0^t e^{8x}\;dx = 2e^{-5t}\left[\left(\frac{1}{8}e^{8x}\right)\Big|_0^t\right] = \ldots = \frac{e^{3t}-e^{-5t}}{4}$$

(c) Suppose $h(t) = (f*g)(t)$, where $f(t) = t^3e^{2t}$ and $g(t) = e^{-5t}\cos(3t)$ .

Using the table, compute the Laplace transform of $h$.

$$\mathfrak{L}\{h(t)\}(s) = \mathfrak{L}\{(f*g)(t)\}(s) = \mathfrak{L}\{f(t)\}(s)\,\mathfrak{L}\{g(t)\}(s)$$
$$= \mathfrak{L}\{t^3e^{2t}\}(s)\cdot\mathfrak{L}\{e^{-5t}\cos(3t)\}(s)$$
$$= \left[\frac{6}{(s-2)^4}\right]\cdot\left[\frac{s+5}{(s+5)^2+9}\right]$$

2. (10 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' - xy' + 8(x^2-1)y = 0$$

(a) Obtain the indicial equation for the ODE at $x_0 = 0$ and its two roots.
(b) Then use all the information available and Theorem 6.3 to say what the two non-trivial linearly independent solutions given by the theorem look like without attempting to obtain the coefficients of the power series.
// To determine $r$, you need the indicial equation at $x_0 = 0$ and its roots. Now the indicial equation is $r(r-1) + p_0r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{-x}{x^2}\right] = -1 \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{8x^2-8}{x^2}\right] = -8.$$

Thus, the indicial equation is $r^2 - 2r - 8 = 0$, with two roots $r_1 = 4$ and $r_2 = -2$. Consequently the two linearly independent solutions provided by Theorem 6.3 look like the following:

$$y_1(x) = |x|^4\sum_{n=0}^{\infty} c_nx^n \quad\text{and}\quad y_2(x) = |x|^{-2}\sum_{n=0}^{\infty} d_nx^n + Cy_1(x)\ln|x|$$

3. (10 pts.) Obtain the general solution to the following linear ODE:

$$x^2y'' + xy' + 4y = 2\ln(x) \quad with \quad x > 0 .$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) + 4w(t) = 2t .$$

The corresponding homogeneous equ.: $w''(t) + 4w(t) = 0$.

The auxiliary equation: $m^2 + 4 = (m + 2i)(m - 2i) = 0$

Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ \cos(2t), \sin(2t) \}$

The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that

$$w_p(t) = \frac{1}{2}t$$

is a particular integral. Consequently, the general solution to the original ODE, the one involving $y$, is

$$y(x) = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)) + \frac{1}{2}\ln(x) \quad for \quad x > 0.$$

---

4. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - 6y' + x^2y = 0.$$

Very neatly obtain the recurrence formula(s) needed to determine the coefficients of $y(x)$. *DO NOT WASTE TIME ATTEMPTING TO GET THE NUMERICAL VALUES OF THE COEFFICIENTS.*

First,

$$\begin{aligned} 0 &= x^2y - 6y' + y'' \\ &= x^2\sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} 6nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= \sum_{n=2}^{\infty} c_{n-2}x^n - \sum_{n=0}^{\infty} 6(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \\ &= (2c_2 - 6c_1)x^0 + (6c_3 - 12c_2)x^1 + \sum_{n=2}^{\infty}\left[(n+2)(n+1)c_{n+2} - 6(n+1)c_{n+1} + c_{n-2}\right]x^n \end{aligned}$$

for all $x$ near zero. From this you can deduce that we have

$$c_2 = 3c_1,$$

$$c_3 = 2c_2, \text{ and}$$

$$c_{n+2} = \frac{6(n+1)c_{n+1} - c_{n-2}}{(n+2)(n+1)} \quad for\ n \geq 2.$$

5. (10 pts.) (a) Suppose that $f(t)$ is defined for $t \geq 0$. What is the definition of the Laplace transform of $f$ in terms of a definite integral??

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$

for all $s$ for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 & , \text{ if } 0 < t < 1 \\ e^{2t} & , \text{ if } 1 \leq t. \end{cases}$$

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^1 0e^{-st}\,dt + \int_1^R e^{2t}e^{-st}\,dt\right]$$

$$= \lim_{R\to\infty}\left[\frac{e^{-(s-2)}}{s-2} - \frac{e^{-R(s-2)}}{s-2}\right] = \frac{e^{-(s-2)}}{s-2} \quad \textit{provided } s > 2.$$

---

6. (10 pts.) Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{2s+12}{s^2+2s+10} \overset{(Work)}{=} \dfrac{2(s+1)+10}{(s+1)^2+3^2}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = 2e^{-t}\cos(3t) + \frac{10}{3}e^{-t}\sin(3t)$$

(b) $F(s) = \dfrac{5s+3}{s^2-6s+9} \overset{(Work)}{=} \dfrac{5}{s-3} + \dfrac{21}{(s-3)^2}$ *using a pfd.*

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = 5e^{3t} + 21te^{3t}$$

'Tis just the usual magic of multiplication by '1' in the correct form or the addition of '0' suitably transmogrified with linearity tossed into the mix.

---

7. (5 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^2-4x)^2y'' + (x-4)y' + (x+1)y = 0$$

An equivalent equation in standard form is

$$y'' + \frac{x-4}{(x(x-4))^2}y' + \frac{x+1}{(x(x-4))^2}y = 0.$$

From this, we can see easily that $x_0 = 0$ is an irregular singular point of the equation, and $x_0 = 4$ is a regular singular point. All other real numbers are ordinary points of the equation.

8. (15 pts.) Using only the Laplace transform machine, solve the following first order initial-value problem.

$$y'(t) - 2y(t) = t^3 e^{2t} \ ; \ \ y(0) = 3 \ .$$

By taking the Laplace Transform of both sides of the differential equation, and using the initial condition, we have

$$\mathcal{L}\{y'(t)\}(s) - 2\mathcal{L}\{y(t)\}(s) = \mathcal{L}\{t^3 e^{2t}\}(s)$$

$$\Rightarrow \quad s\mathcal{L}\{y(t)\}(s) - y(0) - 2\mathcal{L}\{y(t)\}(s) = \frac{3!}{(s-2)^4}$$

$$\Rightarrow \quad (s-2)\mathcal{L}\{y(t)\}(s) = 3 + \frac{3!}{(s-2)^4}$$

$$\Rightarrow \quad \mathcal{L}\{y(t)\}(s) = \frac{3}{s-2} + \frac{3!}{(s-2)^5} .$$

By taking inverse transforms now, we quickly obtain

$$y(t) = 3e^{2t} + \frac{1}{4}t^4 e^{2t}.$$

---

9. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients satisfy the following equations:

$$c_2 = \frac{c_1}{2},$$

$$c_3 = \frac{c_2}{3}, \ and$$

$$c_{n+2} = \frac{(n+1)c_{n+1} - c_{n-2}}{(n+2)(n+1)} \quad for\ n \ge 2 .$$

Determine the exact numerical value of the coefficients $c_0$, $c_1$, $c_2$, $c_3$, and $c_4$ for the particular solution that satisfies the initial conditions $y(0) = -2$ and $y'(0) = 1$.

$c_0 = y(0) = -2$ $\qquad$ $c_1 = y'(0) = 1$

$c_2 = \dfrac{c_1}{2} = \dfrac{1}{2}$ $\qquad$ $c_3 = \dfrac{c_2}{3} = \dfrac{1}{6}$

$c_4 = \dfrac{3c_3 - c_0}{(4)(3)} = \dfrac{5/2}{12} = \dfrac{5}{24}$

---

10 Point Bonus ? : (a) With proof, give an example of an initial-value problem at $x_0 = 0$ with a nonzero solution and the ODE linear homogeneous of 1st order, such that the Laplace transform of the IVP is a linear ODE of 2nd order. (b) Provide the solution to the un-transformed 1st order IVP.

[Say where your work is below, for there isn't room for your work here. See de-t3-bo.pdf. ]