NAME: Em Toidi [BOnus Briefly] TEST3/MAP2302

10 Point Bonus ? : (a) With proof, give an example of an initial-value problem at $x_0 = 0$ with a non-zero solution and the ODE linear homogeneous of 1st order, such that the Laplace transform of the IVP is a linear ODE of 2nd order. (b) Provide the solution to the un-transformed 1st order IVP.

(a) There are, of course, infinitely many examples of such initial-value problems. The key property of the Laplace transform is that involving the transform of a product of an integer power of t with an arbitrary function defined on the non-negative reals that has a Laplace transform. So consider the following IVP:

$$y'(t) + t^2 y(t) = 0 \; ; \; y(0) = 1 \, .$$

Then by taking transforms, we have

$$\mathfrak{L}\{y'(t)\}(s) + \mathfrak{L}\{t^2 y(t)\}(s) = \mathfrak{L}\{0\}(s)$$

$$\Rightarrow \quad s\mathfrak{L}\{y(t)\}(s) + \frac{d^2}{ds^2}\mathfrak{L}\{y(t)\}(s) = -1$$

$$\Rightarrow \quad Y''(s) + sY(s) = -1 \quad \textit{when } Y(s) = \mathfrak{L}\{y(t)\}(s) \, .$$

(b) The linear ODE in Part (a) clearly has

$$\mu(t) = \exp\left(\frac{t^3}{3}\right)$$

as an integrating factor. By using the standard recipe, it follows that the solution to the 1st order IVP is actually

$$y(t) = \exp\left(-\frac{t^3}{3}\right) .$$

Of course you may verify routinely that this function satisfies the IVP.

The reason for this exercise is to give an indication of why we have been restricting our attention to constant coefficient equations in utilizing the Laplace transform for the solution of IVP’s.