NAME: Ogre Ogre

Silly 20 point bonus: It turns out that f(t) = ln(t) for t > 0 has a perfectly good Laplace transform. Although the integral is improper with respect to both limits of integration, by using appropriate comparison tests, one can show the integral converges for s > 0. Accept this as given. $\mathfrak{L}$\{ln(t)\} is a particular solution to a 1st order linear ODE. Obtain $\mathfrak{L}$\{ln(t)\}.

Added Hints: (1) Define the function G by

$$G(t) = \int_0^t \ln(x)dx \text{ for } t > 0, \text{ and } G(0) = 0.$$

Then G′(t) = ln(t) for t > 0. The integral defining G is improper, but has a useful alias when t > 0. (2) For your purposes, the numerical constant may be written in terms of $\mathfrak{L}$\{ln(t)\}(1). Better: Write the numerical constant in terms of the gamma function!!
Where is your work?? Right here, Fred!

Solution: It is clear that the first thing we need to do is produce the mythical ODE that the Laplace transform of the natural log satisfies. Evidently, G′(t) = ln(t) for t > 0 implies that

$$(*) \qquad s\mathfrak{L}\{G(t)\}(s) = \mathfrak{L}\{G'(t)\}(s)$$

$$= \mathfrak{L}\{\ln(t)\}(s) \text{ for } s > 0$$

since G(0) = 0. Now it helps to reveal the alias for G(t) alluded to in the additional hints. Evaluate the improper integral defining G. Since we can see

$$\lim_{x\to 0^+} x\ln(x) = 0$$

by using L'Hopital's Rule after doing a little algebra, it is easy to verify that G(t) = t·ln(t) - t for t > 0. Thus,

$$\mathfrak{L}\{G(t)\}(s) = \mathfrak{L}\{t\cdot\ln(t)\}(s) - \mathfrak{L}\{t\}(s)$$

$$= -\frac{d[\mathfrak{L}\{\ln(t)\}]}{ds}(s) - \frac{1}{s^2}.$$

This means, of course, that the Laplace transform of G above is given in terms of the derivative of the Laplace transform of the natural log. So substitute into the left side of (*). What we then obtain after performing some quite routine algebraic prestidigitation is

$$\frac{d[\mathfrak{L}\{\ln(t)\}]}{ds}(s) + \frac{1}{s}\mathfrak{L}\{\ln(t)\}(s) = -\frac{1}{s^2},$$

for s > 0. Consequently, at least formally, the Laplace transform of the natural log is a solution to a first order linear ODE for s > 0. [Did you notice that the Laplace transform of the natural log is continuous for s > 0 ??]

So what is that differential equation? This:

(**) $$X'(s) + \frac{1}{s}X(s) = -\frac{1}{s^2},$$

with s > 0.

This linear varmint is easy to solve. A fairly obvious integrating factor is $\mu(s) = s$ for s > 0. Just apply the recipe. You will discover that the general solution to (**) is

$$X(s) = -\frac{\ln(s) + C}{s},$$

where C is an arbitrary constant and s > 0.

From the Fundamental Existence - Uniqueness Theorem, in Chapter 1 of Ross, $\mathcal{L}\{\ln(t)\}(s)$ must be one of these solutions.

The key piece is the magic constant C. If

$$\mathcal{L}\{\ln(t)\}(s) = -\frac{\ln(s) + C}{s},$$

then

$$\mathcal{L}\{\ln(t)\}(1) = -\frac{\ln(1) + C}{1} = -C.$$

Thus,

$$C = -\mathcal{L}\{\ln(t)\}(1) = -\int_0^\infty \ln(t)e^{-t}dt.$$

What is a little less obvious is that

$$C = -\Gamma'(1).$$

What is much less obvious, at this level, is the C = γ, the mysterious Euler constant. Do you recall that

$$\gamma = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} - \ln(n) \quad ??$$

[To this day, no one has yet determined, with proof, whether γ is a rational number.]