NAME:

Student Number: Exam Number:

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Since the answer really consists of all the magic transformations and incantations, do not "box" your final results. Show me all the magic on the page.*

1. (120 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation. (20 pts./part)

(a) $y'' + y = \csc(x)$ for $0 < x < \pi$.

(b) $x^2y'' - xy' + y = 4x^2$; $y(1) = 1$ and $y'(1) = 0$.

1. (c) $y' = 3x^2(1 + y^2)$; $y(0) = 3^{1/2}$

1. (d) $y' + x^{-1}y = 12xy^{-2}$ for $x > 0$.

1. (e) $(x^2 + y^2)dx - 4xy \cdot dy = 0 \; ; \; y(1) = 1$

1. (f) $[5y^2 + \sin^2(x)]dx + [10xy - \tan^2(y)]dy = 0$

2.(10 pts.) Work the following problem which uses Hooke's law: [Free motion, but damped.]
//An eight pound weight is attached to the lower end of a coil spring suspended from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 inches. The weight is then pulled down 9 inches below its equilibrium position and released at $t = 0$. The medium offers a resistance in pounds numerically equal to $4x'$, where x' is the instantaneous velocity in feet per second. Determine the displacement of the weight as a function of time.//

3. (15 pts.)
The equation $x^2y'' + 2xy' + (x^2 - 6)y = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved. [Hint: Use ALL the information you have available after solving the indicial equation. Write those power series varmints right carefully folks.]

4.(15 pts.)
It is known that $f(x) = x^r$ is a solution of the homogeneous linear differential equation

(*) $x^2y'' + 3xy' + y = 0$

for a particular value of r.

(a) Find the value of r by substituting f(x) into (*), obtaining an algebraic equation in r, and solving the equation involving r.

(b) Then if you were going to find a second, linearly independent solution to (*) by using only the technique of reduction of order, what is the 1st order linear differential equation you would have to solve?? Produce the 1st order equation and stop.

5. (10 pts.) Work the following problem which uses Newton's Law of Cooling:

// Assume a body of temperature 200°F is placed at time t = 0 in a medium the temperature of which is maintained at 80°F. At the end of ten minutes the temperature of the body has dropped to 180°F. When will the temperature of the body be 100°F?? //

Be sure to state what your variables represent using complete sentences.

6. (15 pts.) Use only the Laplace transform machine to completely solve the following initial value problem.

$$y' - y = f(t) \text{ , where } f(t) = \begin{cases} \sin(t) & \text{, for } 0 \le t < \pi \\ 0 & \text{, for } \pi \le t \end{cases}$$

and $y(0) = 1$.

7. (15 pts.) **On the back of Page 5 of 6:**

(a) Obtain the recurrence formula for the power series solution at $x_0 = 0$ of the homogeneous linear O.D.E. $y'' - xy = 0$.

(b) Compute the first five (5) coefficients of the unique solution $y_1(x)$ that satisfies the initial conditions $y_1(0) = 1$ and $y_1'(0) = 1$.

Silly 20 point bonus: It turns out that $f(t) = \ln(t)$ for $t > 0$ has a perfectly good Laplace transform. Although the integral is improper with respect to both limits of integration, by using appropriate comparison tests, one can show the integral converges for $s > 0$. Accept this as given. $\mathfrak{L}\{\ln(t)\}$ is a particular solution to a 1st order linear ODE. Obtain $\mathfrak{L}\{\ln(t)\}$.

Added Hints: (1) Define the function G by

$$G(t) = \int_0^t \ln(x)dx \text{ for } t > 0, \text{ and } G(0) = 0.$$

Then $G'(t) = \ln(t)$ for $t > 0$. The integral defining G is improper, but has a useful alias when $t > 0$. (2) For your purposes, the numerical constant may be written in terms of $\mathfrak{L}\{\ln(t)\}(1)$. Better: Write the numerical constant in terms of the gamma function!!
Where is your work??