1. (85 pts.) Solve each of the following differential equations or initial value problems. Show all essential work neatly and correctly. [17 points/part]

(a) $3y' + x^{-1}y = 9xy^{-2}$ with $y(1) = 2$. This is obviously a Bernoulli equation. Since the DE is equivalent to the equation

$$3y^2\frac{dy}{dx} + \frac{1}{x}y^3 = 9x \ ,$$

set $v = y^3$. Then $\frac{dv}{dx} = 3y^2\frac{dy}{dx}$. Substituting yields the linear equation

$$\frac{dv}{dx} + \frac{1}{x}v = 9x \ ,$$

which has $\mu = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$ for $x > 0$ as an integrating factor. Multiplying the linear equation by μ yields

$$\frac{d[xv]}{dx} = 9x^2 \ .$$

Integrating both sides, we have $xv = 3x^3 + C$. Replacing v results in a one-parameter family of solutions that is given by $xy^3 = 3x^3 + C$. By using the initial condition, $y(1) = 2$, it follows that $C = 5$. Thus, an implicit solution to the IVP is $xy^3 = 3x^3 + 5$. An explicit solution is given by $y(x) = (3x^2 + 5x^{-1})^{1/3}$ for $x > 0$.

(b) $(ye^{xy} - 6x^2)dx + (xe^{xy} + 8y^3)dy = 0$ Since

$$\frac{\partial}{\partial y}(ye^{xy} - 6x^2) = e^{xy} + xye^{xy} = \frac{\partial}{\partial x}(xe^{xy} + 8y^3) \ ,$$

this equation is exact. Thus, there is a nice function F(x,y) satisfying

$$(1)\ \ \frac{\partial F}{\partial x} = ye^{xy} - 6x^2 \quad \text{and} \quad (2)\ \ \frac{\partial F}{\partial y} = xe^{xy} + 8y^3 \ .$$

It follows from (1) above that we have

$$(3)\ \ F(x,y) = \int ye^{xy} - 6x^2\,dx = e^{xy} - 2x^3 + c(y) \ ,$$

where c(y) is some function that we have yet to determine. Now using (2) and (3),

$$xe^{xy} + 8y^3 = \frac{\partial}{\partial y}(e^{xy} - 2x^3 + c(y)) = xe^{xy} + \frac{d}{dy}c(y) \ ,$$

which implies $\frac{d}{dy}c(y) = 8y^3$. Integrating now tells us $c(y) = 2y^4 + c_0$ for some constant c_0. Thus,

$$F(x,y) = e^{xy} - 2x^3 + 2y^4 + c_0 \ .$$

A one-parameter family of implicit solutions is given by

$e^{xy} - 2x^3 + 2y^4 = c$, where c is an arbitrary constant.

(c)

$$y' + y = f(x) \text{ , where } f(x) = \begin{cases} 6 & \text{, for } 0 \le x < 3 \\ 2x & \text{, for } 3 \le x \end{cases}$$

and $y(0) = -3$.

Clearly the ODE is linear and has $\mu = e^x$ as an integrating factor. We may cope with the piecewise defined function f by dealing with a sequence of initial value problems whose solutions, when glued together, will provide the solution to (c). [We shall leave many of the details to you. They are quite routine. Just tread very carefully.]

(I) $y' + y = 6$ and $y(0) = -3$: Multiplying the ODE by μ, doing the obligatory integration, and determining the constant of integration leads to the explicit solution, $y(x) = 6 - 9e^{-x}$ for x satisfying $0 \le x < 3$. [You really do need the explicit solution here to be able to deal effectively, easily with the initial condition of the next step.]

(II) $y' + y = 2x$ and

$$y(3) = \lim_{x \to 3^-} y(x) = \lim_{x \to 3^-} (6 - 9e^{-x}) = 6 - 9e^{-3} \quad \text{for } 3 \le x \text{ :}$$

Multiplying by μ leads us to $\dfrac{d[e^x y]}{dx} = 2xe^x$. Then integrating by parts and multiplying by e^{-x} yields $y(x) = 2x - 2 + de^{-x}$ for some number d. Using the initial condition, $y(3) = 6 - 9e^{-3}$, it turns out that $d = 2e^3 - 9$, so $y(x) = 2x - 2 + (2e^3 - 9)e^{-x}$ for $3 \le x$.

Thus, the solution to (c) is given by

$$y(x) = \begin{cases} 6 - 9e^{-x} & , 0 \le x < 3 \\ 2x - 2 + (2e^3 - 9)e^{-x} & , 3 \le x \end{cases} .$$

(d) $(4y^2 + 1)dx + (10 \cdot \csc(x))dy = 0$ This is separable as written. Since (d) is equivalent to the equation

$$(4y^2 + 1) + (10 \cdot csc(x))\frac{dy}{dx} = 0$$

and the polynomial $4y^2 + 1$ has no real zeros, we see that there are no constant solutions to be lost by separating variables. Separating variables yields

$$\sin(x)dx + \frac{10}{(2y)^2 + 1}dy = 0.$$

Thus, a one-parameter family of solutions is given by

$$\int \sin(x)dx + \int \frac{10}{(2y)^2 + 1}dy = C$$

or after evaluating the antiderivatives,

$$-\cos(x) + 5 \cdot \tan^{-1}(2y) = C. \quad //$$

You can actually obtain explicit solutions easily here.

(e) $(4x^2 + y^2)dx + (x^2 - xy)dy = 0$ It is not difficult to see that (e) is homogeneous with the coefficient functions homogeneous of degree two. Since

$$\frac{dy}{dx} = -\frac{4x^2+y^2}{x^2-xy} = \frac{4x^2+y^2}{xy-x^2} = \frac{4 + (y/x)^2}{(y/x) - 1},$$

set $v = \frac{y}{x}$ *so that* $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting now

produces $v + x\frac{dv}{dx} = \frac{4 + v^2}{v - 1}$. Multiplying both sides of this

equation by v - 1 and then doing a little additional algebra

produces $(v+4)dx - (x(v-1))dv = 0$, a separable equation. [At this stage, we can actually see that v = -4 is a constant solution to the separable equation that will be lost by separating variables. The corresponding solution to (e): y = -4x.] Separating variables and integrating yields

$$C = \int\frac{1}{x}dx - \int\frac{v-1}{v+4}dv = \ln|x| - \int\frac{(v+4)-5}{v+4}dv = \ln|x| - \int 1 - \frac{5}{v+4}dv.$$

Consequently, $\ln|x| - v + 5\ln|v+4| = C$. A one-parameter family

of implicit solutions is given by $\ln|x| - \frac{y}{x} + 5\ln|\frac{y}{x} + 4| = C$.

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2. (5 pts.) It is known that every solution to the differential equation $y'' + 16y = 0$ is of the form

$$y = c_1\cdot\sin(4x) + c_2\cdot\cos(4x).$$

Which of these functions satisfies the initial conditions $y(\pi/4) = 2$ and $y'(\pi/4) = -4$?? [Hint: Determine c_1 and c_2 by solving an appropriate linear system. Don't waste time verifying y, above, is a solution.]

Since $y(x) = c_1\cdot\sin(4x) + c_2\cdot\cos(4x)$ implies that $y'(x) = 4c_1\cdot\cos(4x) - 4c_2\cdot\sin(4x)$, the initial conditions imply that $c_1 = 1$ and $c_2 = -2$. Thus, the function that satisfies the initial conditions is $y = \sin(4x) - 2\cdot\cos(4x)$. //

3. (10 pts.) (a) It is known that $f(x) = x^r$ is a solution to the ordinary differential equation

$$(*) \quad x^2y'' + 3xy' - 3y = 0$$

for certain values of the constant r. Determine all such values of r.

Evidently $f(x) = x^r$ is a solution to (*) if, and only if

$$0 = -3x^r + 3xrx^{r-1} + x^2r(r-1)x^{r-2}$$

$$= (-3 + 3r + r(r-1))x^r$$

$$= (r^2 + 2r - 3)x^r$$

$$= (r + 3)(r - 1)x^r \quad \text{for every real number } x.$$

This happens precisely when it is true that $r = -3$ or $r = 1$.

(b) If the differential equation

$$(**) \qquad (Ax + By)dx + (Cx + Dy)dy = 0$$

is exact, what must be true about the constants A, B, C, and D ?

If (**) is exact, then

$$B = \frac{\partial}{\partial y}(Ax+By) = \frac{\partial}{\partial x}(Cx+Dy) = C \ .$$

A and D may be arbitrary numbers, totally unrelated.

Silly 10 Point Bonus: Frodo asked Gandalf, "Do you know of a closed form for the power series function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k2^k}(x-4)^k \quad ?$$

I know the function is defined on the interval I = [2,6)." Gandalf stood silent for a few minutes with a furrowed brow and then replied, "Of course. The closed form is an alias for the function f, which is the solution to an initial value problem to which the magical Fundamental Theorem may be applied." Then Gandalf vanished mysteriously after leaving behind a rapidly fading cheshire cat grin.

Help Frodo.

(a) What is the easy to solve IVP ?? (b) Reveal to Frodo the other identity of the function f. // Say where your work is here: