
Read me first: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. Be very careful. Remember that what is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate.*

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1. (40 pts.) Solve each of the following second order differential equations or initial value problems. Be very careful. Show all essential work. Do not write nonsense.

(a) $y'' + 10y' + 21y = 0$

Auxiliary Equation: $m^2 + 10m + 21 = 0$

A Fundamental Set: $\{ e^{-3x}, e^{-7x} \}$

General Solution: $y = c_1e^{-7x} + c_2e^{3x}$

(b) $d^3y/dx^3 + 25(dy/dx) = 0$

Auxiliary Equation: $m^3 + 25m = 0$

A Fundamental Set: $\{ 1, \cos(5x), \sin(5x) \}$

General Solution: $y = c_1 + c_2\sin(5x) + c_3\cos(5x)$

(c) $y'' - 12y' + 36y = 0$

Auxiliary Equation: $m^2 - 12m + 36 = 0$

A Fundamental Set: $\{ e^{6x}, xe^{6x} \}$

General Solution: $y = c_1e^{6x} + c_2xe^{6x}$

(d) $y'' - 2y' = 10$; $y(0) = -2$, $y'(0) = 1$

Corresponding Homogenous Equation: $y'' - 2y' = 0$

Auxiliary Equation: $m^2 - 2m = 0$

A Fundamental Set: $\{ 1, e^{2x} \}$

The driving function, $f(x) = 10$, is a U.C. function. An preliminary U.C. set is $\{ 1 \}$, but since "1" is part of the complementary space, an appropriate U.C. set is $\{ x \}$. Thus, we try $y_p(x) = Ax$, and find out that $A = -5$ by substituting y_p into (d) and equating coefficients. Thus, the general solution is

$$y(x) = c_1 + c_2e^{2x} - 5x.$$

[Using variation of parameters here is silly.]
Now, finally, applying the initial conditions provides us the solution to the I.V.P.: $y(x) = -5 + 3e^{2x} - 5x$.

2. (5 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m^2(m - \pi)^2(m - (1+2i))^3(m - (1-2i))^3 = 0$$

(a) (3 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**] (b) (2 pt.) What is the order of the differential equation?

(a):

$$y = c_1 + c_2x + c_3e^{\pi x} + c_4xe^{\pi x} + c_5e^x\cos(2x) + c_6e^x\sin(2x)$$
$$+ c_7xe^x\cos(2x) + c_8xe^x\sin(2x)$$
$$+ c_9x^2e^x\cos(2x) + c_{10}x^2e^x\sin(2x)$$

(b): The equation is a tenth order ODE. Simply look at the degree of the polynomial making up the left side of the auxiliary equation.

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3. (10 pts.) Given that $f(x) = \sin(3x)$ is a solution of the homogeneous linear O.D.E. $y'' + 9y = 0$, use only the method of reduction of order to find a second, linearly independent solution. [**WARNING: No reduction, no credit!! Show all the steps of this neatly while using notation correctly. The integral formula for the second solution does not suffice.**]

The substitution of $y = v\cdot\sin(3x)$ into the equation and doing a little algebra yields $0 = \sin(3x)v'' + 6\cdot\cos(3x)v'$. Letting $w = v'$, and doing a bit more algebra allows us to obtain $w' + 6\cdot\cot(3x)w = 0$, a linear homogeneous first order equation with integrating factor $\mu = \sin^2(3x)$. By using this appropriately, we get $w = c\cdot\csc^2(3x)$. Thus , by integrating, we obtain $v = -(c/3)\cdot\cot(3x) + d$. Consequently, by setting $c = -3$ and $d = 0$, we obtain $y = \cos(3x)$ ... no surprise, this.

[Of course you could also choose $c = -3$ and $d = 1$ since the function $y = \sin(3x) + \cos(3x)$ is a linearly independent solution. In fact, as long as c is chosen to be nonzero, you will get a linearly independent solution. Just compute the Wronskian.]

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4. (20 pts.) **Using the method of variation of parameters, not the method of undetermined coefficients**, find a particular integral, y_p, for the differential equation

$$y'' - 2y' = 10.$$

[Hint: Read this problem twice and do exactly what is asked to avoid heartbreak!! *Do not obtain* y_p *using the method of undetermined coefficients*.]

Corresponding Homogeneous: $y'' - 2y' = 0$

A fundamental set: $\{1, e^{2x}\}$.

If $y_p = v_1 \cdot 1 + v_2 e^{2x}$, then v_1' and v_2' are solutions to the following system:

$$1 \cdot v_1' + e^{2x} v_2' = 0$$

$$2e^{2x} v_2' = 10.$$

Solving the system yields $v_1' = -5$ and $v_2' = 5e^{-2x}$. Thus, by integrating, we obtain $v_1 = -5x + c$ and $v_2 = -(5/2)e^{-2x} + d$. Consequently, $y_p = v_1 \cdot 1 + v_2 e^{2x} = -5x - (5/2)$, after cleaning up things. [You may 'drop' $-(5/2)$ from y_p. Why??]

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5. (5 pts.) Consider the linear homogeneous ODE

(**) $$y''(x) + \cos(x)y'(x) + x^2 y(x) = 0.$$

Suppose $f_1(x)$, $f_2(x)$, and $f_3(x)$ are three solutions to (**) defined on the real line with

$f_1(1) = 1$ *and* $f_1'(1) = -2$;

$f_2(1) = -3$ *and* $f_2'(1) = 0$; *and*

$f_3(1) = -5$ *and* $f_3'(1) = 10.$

Let A = { f_1, f_2 } , B = { f_1, f_3 } , and C = { f_2, f_3 }. Which of the above sets is a fundamental set of solutions and which is not?? Explain how we know the difference. [Hint: You might go wronskian, but linear independence is the crux.]

The Wronskian for each pair of functions may be computed at x = 1. Look at the initial conditions satisfied by the given functions. These computations follow:

$$W(f_1, f_2)(1) = f_1(1)f_2'(1) - f_2(1)f_1'(1) = 6$$

$$W(f_1, f_3)(1) = f_1(1)f_3'(1) - f_3(1)f_1'(1) = 0$$

$$W(f_2, f_3)(1) = f_2(1)f_3'(1) - f_3(1)f_2'(1) = -30$$

The theory surrounding the Wronskian allows us to deduce that A is a linearly independent set, B is a linearly dependent set, and C is a linearly independent set. So A and C are fundamental sets of solutions and B is not a fundamental set of solutions.

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6. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular solution, y_p, for the O.D.E.

$$y'' - 6y' = 10x^2e^{6x} - 7\sin(2x) - 32x.$$

Do not attempt to actually find the numerical values of the coefficients!! // Since a fundamental set of solutions to the corresponding homogeneous equation consists of $\{ 1, e^{6x} \}$,

$$y_p = Ax^3e^{6x} + Bx^2e^{6x} + Cxe^{6x} + D\cdot\sin(2x) + E\cdot\cos(2x) + Fx^2 + Gx.$$

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7. (10 pts.) (a) Obtain the differential equation and initial condition needed to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. This will probably involve a second equation relating dependent and independent variables.

//A large water tank initially contains 100 gallons of brine in which 40 pounds of salt is dissolved. Starting at time t = 0 minutes, a brine solution containing 2 pounds of salt per gallon flows into the tank at the rate of 5 gallons per minute. The mixture is kept uniform by a mixer which stirs it continuously, and the well-stirred mixture flows out at the same rate. When will the tank have a mixture containing 50 pounds of salt????//

(a): If $x(t)$ denotes the amount of salt, in pounds, in the tank at time t, in minutes, then an initial value problem modeling the situation is this: $x' = 10 - (5/100)x$ with $x(0) = 40$.

(b) The differential equation may be viewed as separable or linear. Consequently, you may use the techniques from Chapter 2 to deal with it and the initial condition. You may actually use Chapter 4 techniques to solve this without integrating. The solution to the I.V.P.: $x(t) = 200 - 160e^{-(1/20)t}$ with $t \geq 0$.

(c) We have fifty pounds of salt in the tank at the time t_0 when $x(t_0) = 50$. Solving this equation yields $t_0 = 20\cdot\ln(16/15)$.

Lagniappe: To get a very rough idea of how large t_0 is without benefit of a hand calculator, we can find a crude upper bound on

$$\ln(16/15) = \int_1^{16/15} t^{-1}\,dt\ .$$

Since $t^{-1} \leq 1$ on the interval [1,16/15], $\ln(16/15) < 1/15$. This means that $t_0 = 20\cdot\ln(16/15) < 20/15 = 4/3$ minutes. This isn't too horrid. 4/3 is roughly 1.333333 ..., repeating. Using the natural log function on the silly calculator yields 1.290770423.

Silly 10 Point Bonus: Evidently, $\{ 1, e^x \}$ is a fundamental set of solutions for the homogeneous linear constant coefficient ODE $y'' - y' = 0$. Is there a linear second order ODE with constant coefficients with $\{ x, 3e^x \}$ as a fundamental set of solutions? Proof?? Say where your work is here: