NAME: Ogre Ogre **TEST2A/MAP2302** **Bonus Noise**

Silly 10 Point Bonus: Evidently, $\{ 1, e^x \}$ is a fundamental set of solutions for the homogeneous linear constant coefficient ODE $y'' - y' = 0$. Is there a linear second order ODE with constant coefficients with $\{ x, 3e^x \}$ as a fundamental set of solutions? Proof?

Kindly note that the problem does not ask whether $\{ x, 3e^x \}$ is a fundamental set of solutions for the ODE $y'' - y' = 0$. It plainly is not, since "x" is not a solution. What is asked is whether $\{ x, 3e^x \}$ is a fundamental set of solutions for *any* linear second order homogeneous ODE with constant coefficients. Also keep in mind we are only considering real coefficients here.

Solution 1: The simplest thing to do here is to pretend that the set $\{ x, 3e^x \}$ is a fundamental set of solutions to some homogeneous second order linear ODE,

(*) $$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

on some interval. By direct substitution into the ODE and doing a little additional algebra, it follows that if $y_1(x) = x$ is a solution, then the coefficient functions p and q satisfy the equation

(1) $$p(x) + x\cdot q(x) = 0$$

on the interval. Likewise, by substituting $y_2(x) = 3e^x$ into the equation and simplifying the algebra, one finds that p and q also satisfy equation

(2) $$p(x) + q(x) = -1$$

on the interval. Solving the linear system consisting of equations (1) and (2) reveals that if $\{ x, 3e^x \}$ is a fundamental solution to (*), then the coefficient functions p and q must satisfy $p(x) = -x/(x - 1)$ and $q(x) = 1/(x - 1)$. Thus, there is no homogeneous constant coefficient ODE for which $\{ x, 3e^x \}$ is a fundamental set of solutions.

Solution 2: The fundamental existence and uniqueness theorem implies that fundamental sets of solutions of any homogeneous linear constant coefficient ODE will consist of solutions defined on the whole real line. Then it follows from the theory in Section 4.6, the optional reading, that if $\{y_1 , y_2\}$ is a fundamental set of solutions to a homogeneous second order constant coefficient ODE, then the Wronskian is never zero, that is, $W(y_1,y_2)(x) \neq 0$ for every real number x. If we compute the Wronskian of $y_1(x) = x$ and $y_2(x) = 3e^x$, however, we find that $W(y_1,y_2)(x) = 3(x - 1)e^x$ and $W(y_1,y_2)(1) = 0$. Thus, there is no second order homogeneous constant coefficient ODE for which $\{ x, 3e^x \}$ is a fundamental set of solutions.

Solution 3: ???