
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.* Test #:

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1. (15 pts.) (a) Suppose that f(t) is defined for $t \geq 0$. What is the definition of the Laplace transform of f, $\mathfrak{L}\{f(t)\}$, in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$ for each s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 2 , & 4 \geq t > 0 \\ -2 , & t > 4 . \end{cases}$$

$$\begin{aligned}
\mathfrak{L}\{f(t)\}(s) &= \int_0^\infty f(t)e^{-st}dt \\
&= \int_0^4 f(t)e^{-st}dt + \int_4^\infty f(t)e^{-st}dt \\
&= \int_0^4 2e^{-st}dt + \int_4^\infty (-2)e^{-st}dt \\
&= \frac{2}{s} - \frac{2e^{-4s}}{s} - \lim_{R\to\infty}\left(\frac{2e^{-4s}}{s} - \frac{2e^{-Rs}}{s}\right) \\
&= \frac{2}{s} - \frac{4e^{-4s}}{s} \quad \text{if } s > 0.
\end{aligned}$$

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2. (10 pts.) Without evaluating any improper integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $f(t) = 8\cdot\cos^2(4t) - 3t^4\cdot e^{7t}$

$\mathfrak{L}\{f(t)\}(s) = 4/s + (4s/(s^2 + 64)) + 3(4!)/(s-7)^5$

Note: The needed trig identity is the following:

$$\cos^2(\theta) = (1/2)(1 + \cos(2\theta)).$$

(b) $g(t) = 10t\cdot\sin(3t)\cdot e^{-2t}$

$$\begin{aligned}
\mathfrak{L}\{g(t)\}(s) &= 10\mathfrak{L}\{ e^{-2t}(t\cdot\sin(3t))\}(s) \\
&= 10\mathfrak{L}\{ t\cdot\sin(3t) \}(s + 2) \\
&= \frac{10[\ 2(3)(s + 2)\ ]}{[(s + 2)^2 + 9\ ]^2}
\end{aligned}$$

This may also be handled using differentiation.

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3. (15 pts.) Very carefully solve the following initial value problem involving an Euler-Cauchy O.D.E.:

$$x^2y''(x) - xy'(x) + y(x) = 10\cdot\ln(x)$$

$$y(1) = 1,\ y'(1) = 0$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the IVP above transforms into the following IVP in $w(t)$:

$$w''(t) - 2w'(t) + w(t) = 10\cdot t$$

$$w(0) = 1,\ w'(0) = 0.$$

The corresponding homogeneous equ.: $w''(t) - 2w'(t) + w(t) = 0$
The auxiliary equation: $(m - 1)^2 = 0$
Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ e^t,\ t\cdot e^t \}$
The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that $w_p(t) = 10t + 20$ is a particular integral. Dealing with the initial conditions now or slightly later finally yields the following: $y(x) = -19x + 9x\cdot\ln(x) + 10\cdot\ln(x) + 20$.

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4. (10 pts.) Transform the given initial value problem into an algebraic equation in $\mathcal{L}\{y\}$ and solve for $\mathcal{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$\text{I.V.P.:}\quad y''(t) - y'(t) - 6y(t) = 4\cdot\sin(3t)\cos(2t)$$

$$y(0) = 1\ ,\ y'(0) = -2$$

By taking the Laplace transform of both sides of the ODE and using linearity, we have

$$\mathcal{L}\{y''(t)\}(s) - \mathcal{L}\{y'(t)\}(s) - 6\mathcal{L}\{y(t)\}(s) = 4\mathcal{L}\{\sin(3t)\cos(2t)\}(s).$$

Consequently,

$$(s^2 - s - 6)\mathcal{L}\{y(t)\} - sy(0) - y'(0) + y(0) = 4\mathcal{L}\{\sin(3t)\cos(2t)\}.$$

Since $\sin(3t)\cos(2t) = (1/2)[\sin(5t) + \sin(t)]$,

$$4\mathcal{L}\{\sin(3t)\cos(2t)\} = [10/(s^2+25)] + [2/(s^2+1)].\quad \text{Therefore,}$$

$$\mathcal{L}\{y(t)\}(s) = \left(\frac{1}{s^2-s-6}\right)\left(s - 3 + \frac{10}{s^2+25} + \frac{2}{s^2+1}\right)$$

Note: You should be able to very quickly derive a trig identity for $\sin(\alpha)\cos(\beta)$ real-time from

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

and

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha).$$

Look!!

5. (10 pts.)
The equation $x^2 \cdot y'' + x(x - 5)y' + 8y = 0$ has a regular singular point at $x_0 = 0$. Theorems 6.2 and 6.3 imply that there is at least one nontrivial solution of the form

$$y_1(x) = |x|^r \sum_{n=0}^{\infty} c_n x^n$$

and that the series converges for each x satisfying $0 < |x| < R$, for some constant $R > 0$. What can you tell me about the exact value of r for the ODE above? [You need not concern yourself with the values of the c_n's.]

To determine the *r* in question, you need the indicial equation for the ODE at $x_0 = 0$ and its roots. Now the indicial equation is given by $r(r-1) + p_0 r + q_0 = 0$, where

$$p_0 = \lim_{x \to 0} x[x(x-5)/x^2] = -5 \quad \text{and} \quad q_0 = \lim_{x \to 0} x^2[(8)/x^2] = 8 .$$

Thus, the indicial equation is $r^2 - 6r + 8 = 0$, with roots $r_1 = 4$ and $r_2 = 2$. Consequently, the *r* in question is $r_1 = 4$. [The solution corresponding to r_2 *may* involve a logarithmic term. You may also obtain the indicial equation using Ross's method.]

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6. (15 pts.) For (a) - (c) below, suppose that $x_0 = 3$ is a regular singular point of a homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$. For each part, given the indicial equation provided, use all the information available and Theorem 6.3 to say what the solutions y_1 and y_2 look like without attempting to obtain the coefficients of the power series involved.

(a) Indicial equation: $(r - (7/3))(r + (8/3)) = 0$

$$y_1(x) = |x-3|^{7/3} \sum_{n=0}^{\infty} c_n (x-3)^n$$

$$y_2(x) = |x-3|^{-8/3} \sum_{n=0}^{\infty} d_n (x-3)^n + C y_1(x) \ln|x-3|$$

(b) Indicial equation: $(r + (1/2))(r - 1) = 0$

$$y_1(x) = |x-3|^{1} \sum_{n=0}^{\infty} c_n (x-3)^n$$

$$y_2(x) = |x-3|^{-1/2} \sum_{n=0}^{\infty} d_n (x-3)^n$$

(c) Indicial equation: $(r + \pi)(r + \pi) = 0$

$$y_1(x) = |x-3|^{-\pi} \sum_{n=0}^{\infty} c_n (x-3)^n$$

$$y_2(x) = |x-3|^{-\pi+1} \sum_{n=0}^{\infty} d_n (x-3)^n + y_1(x) \ln|x-3|$$

7. (10 pts.) Compute $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{4}{(s-1)(s^2+3)}$ After doing a partial fraction decomposition, you should be able to write

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{s+1}{s^2+3}\right\}(t) = \ldots = e^t - \cos(\sqrt{3}t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$$

after using linearity in the obvious way.

(b) $F(s) = \dfrac{8s + 7}{(s-3)^2 + 4}$ Here we need only invoke an appropriate avatar of zero, and squint just a little to write

$$f(t) = \mathcal{L}^{-1}\left\{\frac{8(s-3) + 31}{(s-3)^2 + 4}\right\}(t) = \ldots = 8e^{3t}\cos(2t) + \frac{31}{2}e^{3t}\sin(2t)$$

after using linearity.

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8. (10 pts.) If $y(x) = \sum_{n=0}^{\infty} c_n x^n$ is a solution of the differential equation $y'' - 2xy' = 0$, obtain the recurrence formula for the coefficients of $y(x)$. What are the values of $c_0, c_1, \ldots, c_4$ when $y(x)$ satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$?

$$\begin{aligned}
0 &= -2xy' + y'' = -2x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\
&= -\sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \\
&= 2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - 2nc_n]x^n \\
&\Rightarrow c_2 = 0, \textit{ and for } n \geq 1,\ c_{n+2} = \frac{2nc_n}{(n+2)(n+1)}.
\end{aligned}$$

So $c_0 = y(0) = 1$, $c_1 = y'(0) = -1$, $c_2 = 0$, $c_3 = -1/3$, $c_4 = 0$.

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9. (5 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$(x^5 - 4x^4 + 4x^3)y'' + x^2y' + x(x + 2)y = 0$$

Since $P_1(x) = \dfrac{x^2}{x^3(x-2)^2}$ and $P_2(x) = \dfrac{x(x+2)}{x^3(x-2)^2}$, it is easy to see that $x = 0$ is a regular singular point, and $x = 2$ is an irregular singular point. All other real numbers are ordinary points of the equation.

Under the table magic 10 point bonus: Reveal in gory detail how #17, the Laplace Transform of $f(t) = t^{\alpha}$, is obtained. Where??