NAME: Ogre Ogre **TEST3/MAP2302** **Bonus Business**

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.* Test #:

Under the table magic 10 point bonus: Reveal in gory detail how #17, the Laplace Transform of f(t) = t^{α}, is obtained. Where??

In order to see that

$$\mathcal{L}\{t^{\alpha}\}(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad \textit{for } \alpha > -1,$$

we plainly shall need to work with the definition of the Laplace transform. Evidently

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \int_0^{\infty} t^{\alpha} e^{-st}\,dt \\ &= \lim_{b\to\infty} \int_0^{b} t^{\alpha} e^{-st}\,dt.\end{aligned}$$

Now consider the gamma function, also defined by an improper integral,

$$\begin{aligned}\Gamma(x) &= \int_0^{\infty} e^{-u} u^{x-1}\,du \\ &= \lim_{b\to\infty} \int_0^{b} e^{-u} u^{x-1}\,du.\end{aligned}$$

Formally, to write the Laplace transform in terms of the gamma function, we should do a simple substitution, namely letting u = st so that $s^{-1}du = dt$ when s > 0. Then, at least in a formal sense,

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \int_0^{\infty} t^{\alpha} e^{-st}\,dt \\ &= \lim_{b\to\infty} \int_0^{b} t^{\alpha} e^{-st}\,dt \\ &= \lim_{b\to\infty} \int_0^{bs} \left(\frac{u}{s}\right)^{\alpha} e^{-u} \frac{1}{s}\,du \\ &= \frac{1}{s^{\alpha+1}} \lim_{b\to\infty} \int_0^{bs} e^{-u} u^{(\alpha+1)-1}\,du \\ &= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\end{aligned}$$

if s > 0.

Of course there is a problem here: convergence. If $\alpha \geq 0$, the convergence of the improper integrals is *not* problematical. All is well. When $-1 < \alpha < 0$, however, we do have a wee difficulty. The integrand is actually unbounded in each interval $(0,\varepsilon)$ when $\varepsilon > 0$. The bottom line is that in this case one must deal with two limits: $b \to \infty$ and $a \to 0^{+}$. I'll spare you the details. What turns out to be needed is an appropriate comparison test. Can you give a precise formulation of this piece of magic ?? Can you complete the proof ???