
Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Clearly show me the all the magic on the page.* Test #:

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1. (40 pts.) Without evaluating any integrals and using only the table provided, properties of the Laplace transform, and appropriate function identities, obtain the Laplace transform of each of the functions that follows:

(a) $h(t) = \begin{cases} -5 & , 0 < t < 10 \\ 22 & , 10 < t < 45 \\ -3 & , 45 < t\ . \end{cases}$

$= -5 + 27u_{10}(t) - 25u_{45}(t)$

$$\mathcal{L}\{h(t)\}(s) = -\frac{5}{s} + \frac{27e^{-10s}}{s} - \frac{25e^{-45s}}{s} \text{ for } s > 0.$$

(b) $f(t) = \begin{cases} 8t & , 0 < t < 4 \\ 16 & , 4 < t \end{cases}$

$= 8t + (16 - 8t)u_4(t)$

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \frac{8}{s^2} + \mathcal{L}\{(16-8t)u_4(t)\}(s) \\ &= \frac{8}{s^2} + \mathcal{L}\{g(t-4)u_4(t)\}(s), \text{ where } g(t-4) = 16-8t \\ &= \frac{8}{s^2} + e^{-4s}\mathcal{L}\{g(t)\}(s), \text{ where } g(t) = 16-8(t+4) = -8t-16 \\ &= \frac{8}{s^2} - e^{-4s}\left[\frac{8}{s^2} + \frac{16}{s}\right] \text{ for } s > 0.\end{aligned}$$

(c) $(f*g)(t)$, when $f(t) = 2\cdot e^{8t}\sin(3t)$ and $g(t) = 2t^4$

$$\begin{aligned}\mathcal{L}\{(f*g)(t)\}(s) &= \mathcal{L}\{f(t)\}(s)\cdot\mathcal{L}\{g(t)\}(s) \\ &= \mathcal{L}\{2e^{8t}\sin(3t)\}(s)\cdot\mathcal{L}\{2t^4\}(s) \\ &= \left[\frac{6}{(s-8)^2+9}\right]\cdot\left[\frac{48}{s^5}\right]\end{aligned}$$

1.

(d) $g(t) = t^2 \cdot e^t \cdot \sin(t)$

Hint: Perform a deft prestidigitation with your magic writing wand correctly and one derivative is all that's needed.

$$\begin{aligned}
\mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{e^t(t(t\sin(t)))\}(s) \\
&= \mathcal{L}\{t(t\sin(t))\}(s-1) \\
&= \frac{6(s-1)^2 - 2}{((s-1)^2 + 1)^3}, \\
\textit{since } \mathcal{L}\{t(t\sin(t))\}(s) &= (-1)\frac{d}{ds}[\mathcal{L}\{t\sin(t)\}(s)] \\
&= (-1)\frac{d}{ds}\left[\frac{2s}{(s^2+1)^2}\right] \\
&= \ldots = \left[\frac{6s^2 - 2}{(s^2+1)^3}\right].
\end{aligned}$$

You could also have looked up the Laplace transform of $e^t \cdot \sin(t)$ and differentiated that a couple of times.

2. (10 pts.) (a) The Laplace transform of the following periodic function may be written in terms of a definite integral. Simply express the transform in terms of the appropriate definite integral, but do not attempt to evaluate that definite integral.

$$f(t) = \begin{cases} t^2 & \text{, for } 0 \le t < 2 \\ (t-4)^2 & \text{, for } 2 \le t < 4, \end{cases}$$

and $f(t) = f(t + 4)$ for $t \ge 0$.

$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^4 f(t)e^{-st}dt}{1 - e^{-4s}}$$

(b) The following sum of definite integrals can be realized as the Laplace transform of a certain function $g(t)$ defined for $t \ge 0$. Provide the precise definition of that function g.

$$\int_0^2 t^2 e^{-st}dt + \int_2^4 (t-4)^2 e^{-st}dt = \mathcal{L}\{g(t)\}(s), \quad \text{where}$$

$$g(t) = \begin{cases} t^2 & \text{, for } 0 \le t < 2 \\ (t-4)^2 & \text{, for } 2 \le t < 4, \\ 0 & \text{, for } 4 \le t. \end{cases}$$

Warning: *Do not attempt to evaluate the Laplace transform of g.*

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3. (15 pts.) Suppose that the Laplace transform of the solution to a certain initial value problem involving a linear differential equation with constant coefficients is given by

$$\mathcal{L}\{y(t)\}(s) = \frac{10e^{-\pi s}}{s^2 + 9} + \frac{8s + 49}{(s + 6)^2 + 9}.$$

What's the solution, y(t) , to the IVP??

$$y(t) = \mathcal{L}^{-1}\left\{\frac{10e^{-\pi s}}{s^2 + 9}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{8(s + 6) + 1}{(s + 6)^2 + 9}\right\}(t)$$

$$= (10/3)u_\pi(t)\sin(3(t-\pi)) + 8e^{-6t}\cos(3t) + (1/3)e^{-6t}\sin(3t)$$

after just a little of the usual mathematical magic --- factoring unity correctly, invoking the avatar of zero who transmogrifies ugly toads to princely table forms, and languishing in linearity --- now ho-hummish. You may write y(t) in piecewise-defined form if you are feeling obstreperous.

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4. (15 pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y' = f(t)\text{ , where } f(t) = \begin{cases} 4 & \text{, for } 0 \le t < 2 \\ 2t & \text{, for } 2 \le t \end{cases}$$

and $y(0) = -1$.

First, observe that $f(t) = 4 + 2(t - 2)u_2(t)$ except at $t = 2$. Thus, applying our friendly Laplace transform to both sides of the differential equation, and using the initial condition, we may produce

$$s\mathcal{L}\{y(t)\}(s) + 1 = [4/s] + [2e^{-2s}/(s^2)].$$

Then solving for the Laplace transform of y yields

$$\mathcal{L}\{y(t)\}(s) = [4/(s^2)] + [2e^{-2s}/(s^3)] - [1/s].$$

Thus, after not bowing at all to the partial fraction proprietor, you may write

$$y(t) = 4t + u_2(t)(t-2)^2 - 1$$

Finally, after you march up and down the unit steps a few times, you have

$$y(t) = \begin{cases} 4t - 1 & \text{, for } 0 \le t < 2 \\ t^2 + 3 & \text{, for } 2 \le t \end{cases}$$

more or less. There are, of course, a couple of inequalities that we have fudged.

5. (10 pts.) Using only the definition of the convolution in terms of a definite integral, not some shenanigans involving the Laplace transform, compute (f*g)(t) when f(t) = t and g(t) = e^t. Begin at the equal sign below.

$$
\begin{aligned}
(f*g)(t) &= \int_0^t f(x)g(t-x)\,dx \\
&= \int_0^t xe^{t-x}\,dx \\
&= e^t\int_0^t xe^{-x}\,dx \\
&= e^t\left[(-xe^{-x})\Big|_0^t - \int_0^t (-e^{-x})\,dx\right] \\
&= e^t\left[-te^{-t} + \int_0^t e^{-x}dx\right] \\
&= e^t - 1 - t.
\end{aligned}
$$

6. (10 pts.) Very neatly transform the given initial value problem into a linear system in $\mathfrak{L}\{x\}$ and $\mathfrak{L}\{y\}$ and stop. Do not attempt to solve for $\mathfrak{L}\{x\}$ or $\mathfrak{L}\{y\}$.

I.V.P.: $2x'(t) + y(t) = 12\cdot\delta(t - 8)$

$x(t) - 3y'(t) = 8\cdot e^{8t}$, $x(0) = -4$, $y(0) = 1$

After carefully taking the Laplace transform of both sides of both equations and doing just a little extra algebra, you should obtain a linear system that resembles

$$2s\mathfrak{L}\{x\} + \mathfrak{L}\{y\} = 12e^{-8s} - 8$$

$$\mathfrak{L}\{x\} - 3s\mathfrak{L}\{y\} = 8(s-8)^{-1} - 3.$$

Yes, it is very routine algebra. Just be patient.

10 Point Bonus: What's the exact value of the following definite integral: $\int_0^\infty |\sin(t)|e^{-t}\,dt$ Where's your work??

Hints:
- (a) Yes, this may be viewed in terms of the Laplace transform.
- (b) Yes, those are absolute value bars.
- (c) Yes, $|\sin(t)|$ is π-periodic.
- (d) No, you don't have to do integration by parts, but *tanstaafl* applies. Do you grok the unit stepping dance, the heavy side of thimble rigging??