Student Number: Exam Number:

Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be very careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me all your magic on the page.*

1. (140 pts.) Solve each of the following differential equations or initial value problems. If an initial condition is not given, display the general solution to the differential equation.
(20 pts./part)

(a) $\frac{dy}{dx} = \cos(x)\left(1 + y^2\right)$, $y(0) = \pi/6$.

(b) $\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^3}$, $y(1) = 2\pi$.

1.(c) $(xy^2 + 4\cos(x))dx + (x^2y - 4\sin(y))dy = 0$

(d) $(2xy - x^2)dx - (xy)dy = 0$

1.(e) $\frac{d^2y}{dx^2} + y = \cos(x)$, $y(0) = 1$ and $y'(0) = -1$.

(f) $2\frac{dy}{dx} + \frac{1}{x}y = 8xy^{-1}$

1.(g) $\dfrac{d^2y}{dx^2} + y = \tan(x)$

2. (10 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - x^2 y = 0.$$

(a) Obtain the recurrence formula for the coefficients of $y(x)$.
(b) Which of the coefficients must be zero, no matter what the initial conditions may be?
(c) If $y(x)$ also satisfies the initial conditions $y(0) = 1$ and $y'(0) = 1$, what is the numerical value of c_5 ??

3. (10 pts.) A large water tank initially contains 100 gallons of brine in which 50 pounds of salt is dissolved. Starting at time $t = 0$ minutes, a brine solution containing 4 pounds of salt per gallon flows into the tank at the rate of 5 gallons per minute. The mixture is kept uniform by a mixer which stirs it continuously, and the well-stirred mixture flows out at the same rate. When will the tank have a mixture containing 200 pounds of salt???? Explain. Details are essential here!!

4. (10 pts.)

The equation $x^2 \cdot y'' + x(x + 4)y' - 10y = 0$ has a regular singular point at $x_0 = 0$. Find the indicial equation of this O.D.E. at $x_0 = 0$ and determine its roots. Then, using all the information now available and Theorem 6.3, say what the general solution at $x_0 = 0$ looks like without attempting to obtain the coefficients of the power series functions involved. [Hint: Use ALL the information you have available after solving the indicial equation. Write those power series varmints right carefully folks.]

5. (10 pts.) Work the following problem which uses Hooke's law:

Be sure to state what your variables represent using complete sentences.

// A 256 pound stone is attached to the lower end of a spring with a fixed support. (The spring is vertical.) The weight stretches the spring 1 foot when in its equilibrium position. If the weight is then pushed up 8 inches and released at time $t = 0$ with an initial velocity of 4 inches per second directed downward, obtain the displacement as a function of time. [Assume free, undamped motion.] //

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6. (10 pts.)

It is known that $f(x) = x^r$ is a solution of the homogeneous linear differential equation

(*) $x^2y'' - 5xy' + 9y = 0$

for a particular value of r.

(a) Find the value of r by substituting $f(x)$ into (*), obtaining an algebraic equation in r, and solving the equation involving r.

(b) Then find a second, linearly independent solution to (*) by using the technique of reduction of order.

(c) Using the wronskian, verify that the two functions you get from parts (a) and (b) are indeed linearly independent.

7. (10.pts.) Using only the Laplace transform machine, very carefully solve the following very dinky first order initial value problem:

$$y'(t) = f(t) \text{ and } y(0) = 1,$$

where

$$f(t) = \begin{cases} \cos(t) & , \text{ if } 0 \le t \le 2\pi \\ 0 & , \text{ if } 2\pi < t. \end{cases}$$

Silly 20 point bonus: (a) It's easy to obtain a linear 4th order constant coefficient homogeneous ODE with sin(x) and cos(2x) as solutions. Do so. (b) It's slightly more difficult to obtain a linear 2nd order homogeneous ODE with sin(x) and cos(2x) as solutions. Do this. [Oh, by the way, the equation is NOT a constant coefficient equation.] Say where your work is.