
General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. What is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

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1. (75 pts.) Solve each of the following differential equations or initial value problems. If there is no initial condition, obtain the general solution. [15 points/part]

(a) $\sec(2y)dx + 20(x^2+1)dy = 0$; $y(1) = 2\pi$

First, it is obvious that the DE is separable and that there are no constant solutions since $\sec(2y)$ is never zero. Separating variables provides us with

$$\frac{1}{x^2+1}dx + 20\cos(2y)dy = 0.$$

Integrating this is a snooze and yields

$$\tan^{-1}(x) + 10\sin(2y) = C.$$

The initial condition implies that $C = \pi/4$. Thus, an implicit solution to the IVP is given by

$$\tan^{-1}(x) + 10\sin(2y) = \pi/4.$$

(b) $(y^2 - 4e^{2x})dx + (2xy + \ln(y))dy = 0$

Since

$$\frac{\partial}{\partial y}(y^2 - 4e^{2x}) = 2y = \frac{\partial}{\partial x}(2xy + \ln(y)),$$

the equation is exact. [There are no domain problems since the upper half-plane where $y > 0$, the common domain of the coefficient functions, is simply connected.] Now

$$\frac{\partial F}{\partial x}(x,y) = y^2 - 4e^{2x} \quad \Rightarrow \quad F(x,y) = \int y^2 - 4e^{2x}\,dx = xy^2 - 2e^{2x} + c(y).$$

Consequently,

$$2xy + \ln(y) = \frac{\partial F}{\partial y}(x,y) = 2xy + \frac{dc}{dy}(y) \quad \Rightarrow \quad \frac{dc}{dy}(y) = \ln(y)$$
$$\Rightarrow \quad c(y) = y\ln(y) - y + C_0.$$

Thus,

$$F(x,y) = xy^2 - 2e^{2x} + y\ln(y) - y + C_0.$$

A 1-parameter family of implicit solutions is given by

$$xy^2 - 2e^{2x} + y\ln(y) - y = C_0.$$

(c) $(y^2 + xy + x^2)dx - (x^2)dy = 0$

Although this equation is plainly not exact, it doesn't take a lot of work to see that the coefficient functions are both polynomials that are homogeneous of degree 2. Doing the obvious algebra results in

$$\frac{dy}{dx} = \frac{y^2 + xy + x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) + 1.$$

Setting $v = y/x$ so that $y = vx$, and substituting yields

$$v + x\frac{dv}{dx} = v^2 + v + 1,$$

which, of course, is a separable DE in disguise. Separating variables results in

$$\frac{dv}{v^2 + 1} - \frac{dx}{x} = 0$$

or something similar. Integrating and substituting back for v provides us with $\tan^{-1}(y/x) - \ln(x) = c$ provided $x > 0$. Consequently, a 1 - parameter family of explicit solutions is given by

$$y = x\tan(\ln(x) + c) \quad for\ x > 0.$$

(d) $5\frac{dy}{dx} + \frac{y}{x} = 20x^3y^{-4}$ *with* $x > 0$.

Clearly we have a Bernoulli equation to contend with here. The given DE is equivalent to

$$5y^4\frac{dy}{dx} + \frac{1}{x}y^5 = 20x^3 \quad with \quad x > 0.$$

By setting $v = y^5$ so that

$$\frac{dv}{dx} = 5y^4\frac{dy}{dx},$$

we may transform the Bernoulli equation into the following linear ODE:

$$\frac{dv}{dx} + \frac{1}{x}v = 20x^3 \quad with \quad x > 0.$$

Using the integrating factor $\mu = x$ to solve this linear equation, and substituting back results in a 1 - parameter family of implicit solutions given by $y^5 = 4x^4 + cx^{-1}$ with c an arbitrary constant. [You obviously may solve for y here to get a family of explicit solutions.]

(e) $\dfrac{dr}{d\theta} + \tan(\theta)r = 2\theta\cos(\theta)$; $r(0) = 10$

This varmint is linear all day long. In fact you should already have solved the DE as an assigned homework problem. An integrating factor is easy to come by:

$$\mu(\theta) = e^{\int \tan(\theta)d\theta} = e^{\ln|\sec(\theta)|} = \sec(\theta)$$

for $\theta \in (-\pi/2,\pi/2)$, since we are interested in solutions near zero. Multiplying both sides of the DE by μ results in the following derivative equation:

$$\frac{d}{d\theta}(\sec(\theta)r(\theta)) = 2\theta.$$

By using the Fundamental Theorem of Calculus, we can deal with the initial condition at the same time as we do the integration:

$$\int_0^\theta \frac{d}{dt}(\sec(t)r(t))\,dt = \int_0^\theta 2t\,dt \;\Rightarrow\; \sec(\theta)r(\theta) - \sec(0)r(0) = \theta^2$$
$$\Rightarrow\; r(\theta) = (\theta^2 + 10)\cos(\theta).$$

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2. (10 points) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution to use and perform the reduction, **but do not attempt to solve the separable or homogeneous equation you obtain.**

$$(x - 2y - 3)dx + (2x + y - 1)dy = 0$$

Since the lines defined by the equations $x - 2y - 3 = 0$ and $2x + y - 1 = 0$ intersect at the point (h,k) = (1,-1), the substitution $x = X + 1$ and $y = Y + (-1)$ results in the homogeneous equation

$$(X - 2Y)dX + (2X + Y)dY = 0.$$

Of course to find the point (h,k) = (1,-1), you must solve the system of linear equations.

3. (15 pts.) (a) Obtain the differential equation and initial condition(s) needed to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. [For (c), providing the exact value in terms of a suitable transcendental function will suffice, unless an algebraic function will do the job, in which case use the algebraic function.]

// Assume Newton's Law of Cooling:

A body with temperature of 100 °F is placed at time t = 0 in a medium maintained at a temperature of 20 °F. If, at the end of 10 minutes the temperature of the body is 60 °F, when will the body be 40 °F??

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(a) Let $x(t)$ denote the temperature of the body in °F at time t, in minutes. Then $x' = k(20 - x)$, $x(0) = 100$, and $x(10) = 60$.

[For ease of solution the equation $x(0) = 100$ should be treated as the initial condition, and the equation $x(10) = 60$ should be used to obtain the constant of proportionality, k. Of course, if you are feeling feisty, you might want to try to reverse this!]

(b) The differential equation may be viewed as separable or linear. Since treating it as linear will result in an easy explicit solution, we shall follow the linear brick road.

Here are some of the details of that. The DE may be written as, $x' + kx = 20k$, which is linear with an integrating factor $\mu = e^{kt}$. Using this, results in a general solution to the DE given by $x(t) = 20 + Ce^{-kt}$. Thus, using the I.C. $x(0) = 100$ leads to $x(t) = 20 + 80e^{-kt}$.

(c) By using $x(10) = 60$ now, one can obtain $k = -\ln(1/2)/10$. Thus, $x(t) = 20 + 80(1/2)^{t/10}$. Solving $40 = x(t_0)$ yields $t_0 = 10[(\ln(1/4)/\ln(1/2)] = 10[(\ln(4)/\ln(2)] = 20$ exactly, in minutes, of course, even without a log table.

Bonkers 10 Point Bonus: In an example where he solved a certain homogenous equation, Ross apparently used a table of integrals to evaluate the following integral:

$$\int \frac{1}{(v^2 + 1)^{1/2}}\, dv$$

On the back of Page 3 of 4, with your bare hands reveal all the magic in evaluating this integral. Happy Halloween.