
General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

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1. (15 pts.) Write down the general solution to each of the following linear constant coefficient homogeneous equations.

(a) $y''(x) + 2y'(x) - 8y(x) = 0$

Auxiliary Equation: $m^2 + 2m - 8 = 0$

General Solution: $y = c_1 e^{-4x} + c_2 e^{2x}$

(b) $y''(x) - 6y'(x) + 9y(x) = 0$

Auxiliary Equation: $m^2 - + 6m - 9 = 0$

General Solution: $y = c_1 e^{3x} + c_2 x e^{3x}$

(c) $\dfrac{d^4y}{dx^4} + 9\dfrac{d^2y}{dx^2} = 0$

Auxiliary Equation: $m^4 + 9m^2 = 0$

General Solution: $y = c_1 + c_2 x + c_3 \sin(x) + c_4 \cos(x)$

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2. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$m(m - \pi)^2(m - (1+i))^2(m - (1-i))^2 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**] (b) (5 pt.) What is the order of the differential equation?

$$y = c_1 + c_2 e^{\pi x} + c_3 x e^{\pi x} + c_4 e^{x}\sin(x) + c_5 e^{x}\cos(x) + c_6 x e^{x}\sin(x) + c_7 x e^{x}\cos(x)$$

The order of the differential equation is 7.

3. (10 pts.) It turns out that the nonzero function $f(x) = e^{2x}$ is a solution to the homogeneous linear O.D.E.

$$y'' - 4y = 0.$$

Using only the method of reduction of order, obtain a second, linearly independent solution to this equation.
[**WARNING: No reduction, no credit!! Show all steps of this neatly while using notation correctly. You are being graded on the journey, not the destination.**]

Substitution of $y = ve^{2x}$ into the equation and doing a little algebra yields $0 = v'' + 4v'$. [Observe that you can remove the common exponetial varmint!] Letting $w = v'$, and performing the obvious substitution yields $w' + 4w = 0$, a linear homogeneous first order ODE with the integrating factor $\mu = e^{4x}$. By using this appropriately, we get $w = ce^{-4x}$. Thus $v = -(c/4)e^{-4x} + d$. Consequently, by setting $c = -4$ and $d = 0$, we obtain $y = e^{2x}$ ... no surprise, this. In fact, as long as $c \neq 0$, you will obtain a solution with f and y linearly independent. [Go compute the Wronskian!!]

4. (15 pts.) Very carefully obtain the general solution to the following Euler-Cauchy O.D.E. for x > 0.

$$x^2y''(x) + xy'(x) - y = x^4.$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) - w(t) = e^{4t}.$$

The corresponding homogeneous equ.: $w''(t) - w(t) = 0$.

The auxiliary equation: $(m - 1)(m + 1) = 0$

Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{ e^{t}, e^{-t} \}$

The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that

$$w_p(t) = \frac{1}{15}e^{4t}$$

is a particular integral. Consequently, the general solution to the original ODE, the one involving y, is

$$y(x) = c_1x + c_2x^{-1} + \frac{1}{15}x^4.$$

5. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' + y = 4x - 2x\sin(x) + 7xe^x.$$

First, the corresponding homogeneous equation is

$$y'' + y = 0.$$

which has an auxiliarly equation given by $0 = m^2 + 1$. Thus a fundamental set of solutions for the corresponding homogeneous equation is { $\sin(x)$, $\cos(x)$ }. Taking this into account, we may now write

$$y_p(x) = A + Bx + Cx\sin(x) + Dx\cos(x) + Ex^2\sin(x) + Fx^2\cos(x) + Ge^x + Hxe^x$$

or something equivalent. Here is how the UC sets get built:

(a) For "$4x$" we need $\{1, x\}$. This is OK.
(b) For "$2x \cdot \sin(x)$" we need

$$\{\sin(x), \cos(x), x\cdot\sin(x), x\cdot\cos(x)\}.$$

Unfortunately, the space generated by these functions overlaps the complementary solution space in a nontrivial way. Consequently, we must use a modification of the set above:

$$\{x\cdot\sin(x), x\cdot\cos(x), x^2\sin(x), x^2\cos(x)\}.$$

(c) For "$7xe^x$" we need { e^x, xe^x }. This is OK.

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6. (15 pts.) **Using the method of variation of parameters, not the method of undetermined coefficients**, find a particular integral, y_p, of the differential equation

$$y'' + y' = 4x$$

[**Hint: Read this problem twice and do exactly what is asked to avoid heartbreak!!** *Do not obtain* y_p *using the method of undetermined coefficients. Do not waste time getting the general solution*.]

Corresponding Homogeneous: $y'' + y' = 0$. F.S. = $\{1, e^{-x}\}$.

If $y_p = v_1 1 + v_2 e^{-x}$ then v_1' and v_2' are solutions to the following system:

$$\begin{cases} v_1' + e^{-x}v_2' = 0 \\ \quad -e^{-x}v_2' = 4x \end{cases}$$

Solving the system yields $v_1' = 4x$ and $v_2' = -4xe^x$. Thus, by integrating, we obtain $v_1 = 2x^2 + c$ and $v_2 = -4xe^x + 4e^x + d$. Thus,

$$y_p = v_1 + v_2 e^{-x} = 2x^2 + [((-4x) + 4)e^x]e^{-x} = 2x^2 - 4x + 4.$$

[You may actually drop the constant "4" at the end. Why??]

7. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - 2xy' = 0.$$

(a) Obtain the recurrence formula for the coefficients of $y(x)$.
(b) Which of the coefficients must be zero, no matter what the initial conditions may be?
(c) If $y(x)$ also satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$, what is the numerical value of c_5 ??

(a): First,

$$\begin{aligned} 0 &= -2xy' + y'' \\ &= -2x\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= \sum_{n=1}^{\infty} (-2n)c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \\ &= 2c_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - 2nc_n]x^n. \end{aligned}$$

From this you can deduce that $c_2 = 0$, and that for $n \geq 1$, we have

$$c_{n+2} = \frac{2nc_n}{(n+2)(n+1)}.$$

(b): Since $c_2 = 0$, by using the recurrence formula above, one can safely guess that $c_{2n} = 0$ for $n \geq 1$.

(c):

$$c_0 = y(0) = 0,\ c_1 = y'(0) = 1,\ c_2 = 0,\ c_3 = 1/3,\ c_4 = 0, \text{and } c_5 = 1/10.$$

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8. (10 pts.) Obtain the unique solution to the following initial value problem:

$$y''(x) - y'(x) = 4$$

$$\text{with}$$

$$y(0) = 2 \text{ and } y'(0) = -1.$$

The corresponding homogeneous equation: $y''(x) - y'(x) = 0.$

The auxiliary equation: $0 = m^2 - m = m(m-1)$.
F.S = $\{ 1, e^x \}$. The driving function is an easy UC varmint. Set $y_p = Ax$. Substituting y_p into the original ODE reveals A = -4.

General Solution: $y(x) = c_1 + c_2 e^x - 4x.$

Solution to the IVP: $y(x) = -1 + 3e^x - 4x.$

Silly 10 Point Bonus: Obtain a linear second order ODE with a fundamental set of solutions given by $\{ e^x , 1/x \}$ Say where your work is!