General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

Silly 10 Point Bonus: Obtain a linear second order ODE with a fundamental set of solutions given by { e^x , $1/x$ } Say where your work is!

If there is a homogeneous linear second order ODE with { e^x , $1/x$ } as a fundamental set of solutions, one should expect to be able to determine the coefficient functions, and one should expect that those coefficient functions not be constants. How shall we proceed??

First, for definiteness, assume that the two functions are solutions to

$$(*) \qquad y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

where p and q are functions whose identity we should hope to reveal.

Then, by substituting $f(x) = e^x$ into (*) and simplifying the algebra, we obtain

$$(1) \qquad p(x) + q(x) = -1.$$

Next, doing the same sort of thing thing with $g(x) = x^{-1}$, we obtain

$$(2) \qquad xp(x) - x^2q(x) = 2.$$

What the two equations, (1) and (2), provide is a linear system that p and q must satisfy if f and g are solutions to (*). Solving this little linear system yields

$$p(x) = \frac{2-x^2}{x^2+x} \quad and \quad q(x) = -\frac{2+x}{x^2+x}.$$

It is not difficult to use the Wronskian to see that f and g are linearly independent on the real line with 0 and -1 removed. Direct substitution will reveal that both f and g above are solutions to the differential equation

$$y''(x) + \frac{2-x^2}{x^2+x}y'(x) - \frac{2+x}{x^2+x}y(x) = 0.$$