General directions: *Read each problem carefully and do exactly what is requested. Full credit will be awarded only if you show all your work neatly, and it is correct. Use complete sentences and use notation correctly. What is illegible or incomprehensible is worthless. Since the answer really consists of all the magic transformations, do not box your final result. Show me all the magic on the page. Communicate. Eschew obfuscation.*

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1. (90 pts.) Solve each of the following differential equations or initial value problems. Show all essential work neatly and correctly. [15 points/part]

(a) $\dfrac{dy}{dx} + \dfrac{3}{x}y = \dfrac{48}{x^2}$; $y(1) = 26$

This is linear as written with an integrating factor of $\mu = x^3$. If you carefully follow the recipe and deal with the initial condition, you will get $y(x) = 24x^{-1} + 2x^{-3}$.

(b) $\left(y(xy)^3 - 4e^{2x}\right)dx + \left(x(xy)^3 + 3e^{3y}\right)dy = 0$

This is exact. [**You should check for exactness first!!**] Here's a one-parameter family of solutions: $(1/4)\cdot(xy)^4 - 2e^{2x} + e^{3y} = C$

(c) $\left(y^2 + xy - x^2\right)dx - \left(x^2\right)dy = 0$

This is a homogeneous equation with a routine integration. [The degree of homogeneity is 2.] With $y = vx$, you must deal with the first integral of

$$\int \frac{1}{v^2-1}\,dv - \int \frac{1}{x}\,dx = C.$$

If you must, you can use a trigonometric substitution on the first integral, say $v = \sec(\theta)$, but a partial fraction decomposition is far easier. It turns out that we have

$$(1/2)\ln(|(y - x)/(y + x)| - \ln|x| = c$$

as a one-parameter family of solutions. It turns out that $y = x$ and $y = -x$ are also solutions, but they are not family members. Where did they originate?

(d) $2\cos^2(2y)\,dx + 10\sec(x)\,dy = 0$

This is separable as written. Look. Because (d) is equivalent to $2\cos^2(2y) + 5\sec(x)(dy/dx) = 0$, you get a set of constant solutions from the zeros of cos(2y): $y(x) = (2k+1)(\pi/4)$, k any integer. By separating variables, cleaning up the algebra, and integrating, you get $2\sin(x) + 5\tan(2y) = C$ or ....

(e) $7\frac{dy}{dx} + \frac{1}{x}y = \frac{24x}{y^6}$ for $x > 0$.

This is clearly a Bernoulli equation. Turn it into a linear equation using the substitution $v = y^7$. Yadda, yadda, yadda. $y^7 = 8x^2 + cx^{-1}$ or $y = (8x^2 + cx^{-1})^{1/7}$.

(f) $y'(x) + y(x) = f(x)$, where

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 3 \\ 3, & \text{if } 3 \le x. \end{cases}$$

and $y(0) = 0$. Linear ... with an integrating factor $\mu = e^x$ ... ugh; so gluing the pieces together, we get

$$y(x) = \begin{cases} x - 1 + e^{-x}, & \text{if } 0 \le x < 3 \\ 3 + (1 - e^3)e^{-x}, & \text{if } 3 \le x. \end{cases}$$

[See The TestTomb, Spring 2004 for details of how this is done in a similar problem.]

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2. (5 pts.) It is known that every solution to the differential equation $y'' + y = 0$ is of the form

$$y(x) = c_1\sin(x) + c_2\cos(x).$$

Which of these functions satisfies the initial conditions $y(\pi/2) = 2$ and $y'(\pi/2) = 8$?? From the form of the solution, and the initial conditions, you get a linear system in c_1 and c_2:

$$2 = y(\pi/2) = c_1 \quad \text{and} \quad 8 = y'(\pi/2) = -c_2.$$

Solving the linear system and then replacing c_1 and c_2 in the formula for y yields $y(x) = 2 \cdot \sin(x) - 8 \cdot \cos(x)$.

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3. (5 pts.) For certain values of the constant m the function $f(x) = x^m$ is a solution to the differential equation

$$x^2y''(x) - 6y(x) = 0.$$

Determine all such values of m.

f(x) is a solution if, and only if

$$\begin{aligned} 0 &= x^2f''(x) - 6f(x) \\ &= x^2 \cdot m(m-1)x^{m-2} - 6x^m \\ &= (m^2 - m - 6)x^m \end{aligned}$$

is true for all real numbers x. Hence, $(m-3)(m+2) = 0$. Thus $m = 3$ or $m = -2$.

Bonkers 10 Point Bonus: (a) The Fundamental Theorem of Calculus provides a neat formal solution involving a definite integral with respect to the variable 't' to the following dinky IVP:

$$y'(x) = \sin(x^2) \text{ and } y(0) = 1.$$

What is that solution? (b) Unfortunately $g(x) = \sin(x^2)$ cannot be integrated in elementary terms. Use the answer to (a), the Maclaurin series for sin(x), and term-wise integration, to obtain a power series solution to the IVP *using sigma notation*. [Say where your work is! You don't have room here!]

(a)

$$y(x) = 1 + \int_0^x \sin(t^2)\,dt \text{ for all } x.$$

(b)

$$\begin{aligned} y(x) &= 1 + \int_0^x \sin(t^2)\,dt \\ &= 1 + \int_0^x \sum_{k=0}^{\infty} \left[\frac{(-1)^k (t^2)^{2k+1}}{(2k+1)!}\right] dt \\ &= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k (t^2)^{2k+1}}{(2k+1)!}\,dt \\ &= 1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^x t^{4k+2}\,dt \\ &= 1 + \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)(2k+1)!} \text{ for all } x. \end{aligned}$$