General directions: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Since the answer really consists of all the magic transformations, do not "box" your final results. Communicate. Show me all the magic on the page.*

1. (40 pts.) Solve each of the following second order differential equations or initial value problems. Be very careful. Show all essential work. Do not write nonsense.

(a) $y''(x) - y'(x) - 12y(x) = 0$

Auxiliary Equation: $m^2 - m - 12 = (m-4)(m+3) = 0$

Roots of A.E.: $m = 4$ *or* $m = -3$.

General Solution: $y = c_1 e^{4x} + c_2 e^{-3x}$

(b) $y''(x) - 2y'(x) + 10y(x) = 0$

Auxiliary Equation: $m^2 - 2m + 10 = 0$

Roots of A.E.: $m = 1 + 3i$ *or* $m = 1 - 3i$.

General Solution: $y = c_1 e^x \cos(3x) + c_2 e^x \sin(3x)$

(c) $\dfrac{d^5y}{dx^5} - \dfrac{d^3y}{dx^3} = 0$

Auxiliary Equation: $m^5 - m^3 = m^3(m+1)(m-1) = 0$

Roots of A.E.: $m = 0$ *with multiplicity* 3 *or* $m = 1$ *or* $m = -1$

General Solution: $y = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$

(d) $y''(x) - y'(x) = 1 + x$; $y(0) = 2$, $y'(0) = -1$

The corresponding homogeneous equ.: $y''(x) - y'(x) = 0$.
The auxiliary equation: $(m)(m - 1) = 0$

Fundamental Set = $\{ 1, e^x \}$.

The driving function is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that

$$y_p(x) = -2x - (1/2)x^2$$

General Solution: $y(x) = c_1 + c_2 e^x - 2x - (1/2)x^2$.

Solution to the I.V.P.: $y(x) = 1 + e^x - 2x - (1/2)x^2$.

---

2. (10 pts.) The factored auxiliary equation of a certain homogeneous linear O.D.E. with real constant coefficients is as follows:

$$(m - 2)^2(m - 2i)^3(m + 2i)^3 = 0$$

(a) (5 pts.) Write down the general solution to the differential equation. [**WARNING: Be very careful. This will be graded Right or Wrong!!**] (b) (5 pt.) What is the order of the differential equation?

$$y = c_1e^{2x} + c_2xe^{2x} + c_3\cos(2x) + c_4\sin(2x) + c_5x\cos(2x) + c_6x\sin(2x) + c_7x^2\cos(2x) + c_8x^2\sin(2x)$$

(b): The order of the differential equation is 8.

---

3. (10 pts.) It turns out that the nonzero function f(x) = x is a solution to the homogeneous linear O.D.E.

$$x^2y'' + xy' - y = 0.$$

Using only the method of reduction of order, show how to obtain a second, linearly independent solution to this equation.

[**WARNING: No reduction, no credit!! Show all steps of this neatly while using notation correctly. You are being graded on the journey, not the destination.**]

Substitution of $y = vx$ into the equation and doing a little algebra yields $0 = v'' + (3/x)v'$. Letting $w = v'$, and performing the obvious substitution yields $w' + (3/x)w = 0$, a linear homogeneous first order ODE with the integrating factor $\mu = x^3$. By using this appropriately, we get $w = cx^{-3}$. Thus $v = -(c/2)x^{-2} + d$. Consequently, by setting $c = -2$ and $d = 0$, we obtain $y = x^{-1}$ ... no surprise, this. In fact, as long as $c \neq 0$, you will obtain a solution with f and y linearly independent. [Go compute the Wronskian!!]

---

4. (15 pts.) **Using only the method of variation of parameters, not the method of undetermined coefficients**, reveal how to find a particular integral, y_p, of the differential equation

$$y'' - y = 1$$

[**Hint: Read this problem twice and do exactly what is asked to avoid heartbreak!!** *Do not obtain* y_p *using the method of undetermined coefficients. Do not waste time getting the general solution.*]

Corresponding Homogeneous: $y'' - y = 0$. F.S. = { e^x, e^{-x}}.
If $y_p = v_1e^x + v_2e^{-x}$, then v_1' and v_2' are solutions to the following system:

$$\begin{cases} e^x v_1' + e^{-x}v_2' = 0 \\ e^x v_1' - e^{-x}v_2' = 1 \end{cases}$$

Solving the system yields $v_1' = (1/2)e^{-x}$ and $v_2' = -(1/2)e^{x}$. Thus, by integrating, we obtain $v_1 = -(1/2)e^{-x} + c$ and $v_2 = -(1/2)e^{-x} + d$.
Thus,

$$y_p = v_1e^x + v_2e^{-x} = -\frac{1}{2}e^{-x}e^x + (-1)\frac{1}{2}e^xe^{-x} = -1.$$

[This is known as finding minus one the hard way, 'cause U can't C.]

---

5. (10 pts.) Set up the correct linear combination of undetermined coefficient functions you would use to find a particular integral, y_p, of the O.D.E.

$$y'' + y' = 2x + xe^{-x} + \cos(x)e^{-x}.$$

[**Warning:** *(a) If you skip a critical initial step, you will get no credit!! (b) Do not waste time attempting to find the numerical values of the coefficients!!*]

The corresponding homogeneous equation is

$$y'' + y' = 0.$$

Since a fundamental set of solutions to the corresponding homogeneous equation consists of $\{1,e^{-x}\}$,

$$y_p(x) = Ax + Bx^2 + Cxe^{-x} + Dx^2e^{-x} + E\cos(x)e^{-x} + F\sin(x)e^{-x} .$$

6. (5 pts.) The following differential equation may be solved by either performing a substitution to reduce it to a separable equation or by performing a different substitution to reduce it to a homogeneous equation. Display the substitution and perform the reduction, but **do not attempt to solve the separable or homogeneous equation you obtain**.

$$(x - y - 1)dx + (2x - 2y + 4)dy = 0$$

To determine the substitution, we consider the following system of linear equations:

$$\begin{cases} x - y - 1 = 0 \\ 2x - 2y + 4 = 0. \end{cases}$$

Clearly the lines defined by these equations are parallel. Thus, we set $z = x - y$ so that $dy = dx - dz$. Substituting and doing a tiny bit of algebra yields

$$(3z + 3)dx - (2z + 4)dz = 0,$$

an equation that is plainly *separable*.

7. (10 pts.) (a) Obtain the differential equation and initial condition needed to solve the following word problem. State what your variables represent using complete sentences. (b) Next, solve the initial value problem. (c) Then, answer the last part of the question. [For (c), the exact value in terms of natural logs will suffice.]

//A large water tank initially contains 100 gallons of brine in which 40 pounds of salt is dissolved. Starting at time t = 0 minutes, a brine solution containing 4 pounds of salt per gallon flows into the tank at the rate of 5 gallons per minute. The mixture is kept uniform by a mixer which stirs it continuously, and the well-stirred mixture flows out at the same rate. When will the tank have a mixture containing 50 pounds of salt????//

(a): If $x(t)$ denotes the amount of salt, in pounds, in the tank at time t, in minutes, then an initial value problem modeling the situation is this:

$$x' = 20 - (5/100)x \text{ with } x(0) = 40.$$

(b) The differential equation may be viewed as separable or linear. Consequently, you may use the techniques from Chapter 2 to deal with it and the initial condition. You may also solve this and do no integrations at all once you realize that the equation is, in fact, a constant coefficient linear differential equation with an undetermined coefficient driving function.

Here is the solution to the I.V.P.:

$$x(t) = 400 - 360e^{-(1/20)t} \text{ with } t \geq 0.$$

(c) We have fifty pounds of salt in the tank at the time t_0 when $x(t_0) = 50$. Solving this equation yields $t_0 = 20 \cdot \ln(36/35)$, in minutes, of course.

Silly 10 Point Bonus: What linear homogeneous ODE with constant coefficients has a fundamental set of solutions given by $\{ e^x, xe^x, x^2e^x, x^3e^x \}$ Say where your work is!

Too Easy: The auxiliary equation must be equivalent to $(m - 1)^4 = 0$. Expanding the left hand side of this this will show us the form of the differential equation:

$$y'''' - 4y''' + 6y'' - 4y' + y = 0$$

Use the binomial theorem and Pascal's triangle to do the lifting.