Read Me First: *Show all essential work very neatly. Use correct notation when presenting your computations and arguments. Write using complete sentences. Be careful. Remember this: "=" denotes "equals" , "⇒" denotes "implies" , and "⇔" denotes "is equivalent to". Do not "box" your answers. Communicate. Show me the all magic on the page.*

1. (7 pts.) Locate and classify the singular points of the following second order homogeneous O.D.E. Use complete sentences to describe the type of points and where they occur.

$$((x-2)^2(x-1)^2)y'' + 4x(x-2)y' + (x-1)y = 0$$

Normalized form of the ODE:

$$y'' + \frac{4x(x-2)}{(x-2)^2(x-1)^2}\,y' + \frac{x-1}{(x-2)^2(x-1)^2}\,y = 0.$$

Singular Points: $x_0 = 1$ and $x_0 = 2$.

With a little additional work we can easily see that $x_0 = 2$ is a regular singular point and $x_0 = 1$ is an irregular singular point of the ODE.

2. (18 pts.) For parts (a), (b), and (c) below pretend that $x_0 = 2$ is a regular singular point for some homogeneous linear differential equation of the form $y''(x) + P_1(x)y'(x) + P_2(x)y(x) = 0$, with the ODE actually being different for each part. For each part, given the indicial equation at $x_0 = 2$ provided, use all the information available and Theorem 6.3 to say what the two nontrivial linearly independent solutions look like without attempting to obtain the coefficients of the power series involved.

(a) Indicial equation: (r - (1/3))(r - (1/2)) = 0

$$y_1(x) = |x-2|^{1/2}\sum_{n=0}^{\infty} c_n(x-2)^n$$

$$y_2(x) = |x-2|^{1/3}\sum_{n=0}^{\infty} d_n(x-2)^n \quad \text{since} \quad r_1 = \frac{1}{2} \text{ and } r_2 = \frac{1}{3}, \text{ so } r_1 - r_2 = \frac{1}{6}.$$

(b) Indicial equation: (r - (1/2))(r - (1/2)) = 0

$$y_1(x) = |x-2|^{1/2}\sum_{n=0}^{\infty} c_n(x-2)^n$$

$$y_2(x) = |x-2|^{3/2}\sum_{n=0}^{\infty} d_n(x-2)^n + y_1(x)\ln|x-2| \quad \text{since}$$

$$r_1 = r_2 = \frac{1}{2}, \text{ so } r_1 - r_2 = 0.$$

(c) Indicial equation: (r - (3/2))(r + (7/2)) = 0

$$y_1(x) = |x-2|^{3/2}\sum_{n=0}^{\infty} c_n(x-1)^n$$

$$y_2(x) = |x-2|^{-7/2}\sum_{n=0}^{\infty} d_n(x-2)^n + C\,y_1(x)\ln|x-2| \quad \text{since}$$

$$r_1 = \frac{3}{2} \text{ and } r_2 = -\frac{7}{2}, \text{ so } r_1 - r_2 = 5.$$

3. (15 pts.) Very carefully obtain the solution to the following initial value problem that involves an Euler-Cauchy equation.

$$(*)\begin{cases} x^2y''(x) + xy'(x) + 4y = 4\ln(x) + 8 \ ; \\ y(1) = 1 \ , \ \ y'(1) = -1 \ \ \textit{with } x > 0. \end{cases}$$

By letting $x = e^t$, and $w(t) = y(e^t)$, so that $y(x) = w(\ln(x))$ for $x > 0$, the ODE above transforms into the following ODE in $w(t)$:

$$w''(t) + 4w(t) = 4t + 8.$$

The corresponding homogeneous equ.: $w''(t) + 4w(t) = 0.$

The auxiliary equation: $m^2 + 4 = (m + 2i)(m - 2i) = 0$

Here's a fundamental set of solutions for the corresponding homogeneous equation: $\{\cos(2t), \sin(2t)\}$

The driving function of the transformed equation is a U.C. function. By muttering the appropriate incantation and waving your magic writing utensil over the exam, you find that

$$w_p(t) = t + 2$$

is a particular integral. Consequently, the general solution to the original ODE, the one involving y, is

$$y(x) = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)) + \ln(x) + 2 \ \ \textit{for} \ \ x > 0.$$

By using the two initial conditions now, you can obtain an easy to solve linear system involving the two constants. Solving the system reveals that the solution to the initial value problem is

$$y(x) = -\cos(2\ln(x)) - \sin(2\ln(x)) + \ln(x) + 2.$$

Note: You could also get the constants much more easily by dealing with the transformed initial conditions, $w(0) = 1$ and $w'(0) = -1$ somewhat earlier!!

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4. (10 pts.) (a) Suppose that f(t) is defined for t > 0. What is the definition of the Laplace transform of f, $\mathfrak{L}\{f(t)\}$, in terms of a definite integral??

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\int_0^R f(t)e^{-st}\,dt$$

for all s for which the integral converges.

(b) Using only the definition, not the table, compute the Laplace transform of

$$f(t) = \begin{cases} 0 \ , & \textit{if} \ \ 0 < t < 4 \\ 1 \ , & \textit{if} \ \ 4 < t. \end{cases}$$

$$\mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{R\to\infty}\left[\int_0^4 0e^{-st}dt + \int_4^R e^{-st}dt\right]$$

$$= \lim_{R\to\infty}\left[\frac{e^{-4s}}{s} - \frac{e^{-Rs}}{s}\right] = \frac{e^{-4s}}{s} \ \ \textit{provided } s > 0.$$

Note: You may, of course, check your "answer" using #15 in the table.

5. (15 pts.) Suppose

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

is a solution of the homogeneous second order linear equation

$$y'' - y' + 2xy = 0.$$

Obtain the recurrence formula(s) for the coefficients of $y(x)$.

First,

$$\begin{aligned} 0 &= 2xy - y' + y'' \\ &= 2x\sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &= \sum_{n=1}^{\infty} 2c_{n-1}x^n - \sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n \\ &= (2c_2 - c_1)x^0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - (n+1)c_{n+1} + 2c_{n-1}]x^n. \end{aligned}$$

From this you can deduce that $c_2 = c_1/2$, and that for $n \geq 1$, we have

$$c_{n+2} = \frac{(n+1)c_{n+1} - 2c_{n-1}}{(n+2)(n+1)}.$$

6. (10 pts.)

Compute $f(t) = \mathfrak{L}^{-1}\{F(s)\}(t)$ when

(a) $F(s) = \dfrac{4}{s^4} + \dfrac{3s+4}{s^2+3}$

$$\mathfrak{L}^{-1}\{F(s)\}(t) = \frac{4}{3!}t^3 + 3\cos(\sqrt{3}\,t) + \frac{4}{\sqrt{3}}\sin(\sqrt{3}\,t)$$

(b) $F(s) = \dfrac{8s + 7}{(s-2)^2 + 9}$

$$\mathfrak{L}^{-1}\{F(s)\}(t) = 8e^{2t}\cos(3t) + \frac{23}{3}e^{2t}\sin(3t)$$

Here, of course, linearity and the usual prestidigitation involving zero and one play a role.

7. (5 pts.) The equation below has a regular singular point at $x_0 = 0$.

$$x^2y'' + 2xy' + (x^2 - 6)y = 0$$

Obtain the indicial equation at $x_0 = 0$, and determine its roots.

An equivalent, normalized, ODE is

$$y'' + \frac{2x}{x^2}y' + \frac{x^2-6}{x^2}y = 0.$$

The indicial equation is $r(r\text{-}1) + p_0r + q_0 = 0$ where

$$p_0 = \lim_{x\to 0} x\left[\frac{2x}{x^2}\right] = 2 \text{ and } q_0 = \lim_{x\to 0} x^2\left[\frac{x^2-6}{x^2}\right] = -6.$$

So the indicial equation is $(r - 2)(r + 3) = 0$ with roots 2 and -3.

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8. (10 pts.) The solution to a certain linear ordinary differential equation with coefficient functions that are analytic at $x_0 = 0$ is of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

where the coefficients for $n \geq 2$ satisfy the following equations:

$$(n+2)(n+1)c_{n+2} - n(n+1)c_{n+1} + c_n = 0 \;\; \textit{for all } n \geq 1, \;\; \textit{and} \;\; c_2 = -\frac{1}{2}c_0.$$

Determine the coefficients c_0, c_1, c_2, c_3, and c_4 for the particular solution that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Here is a more user friendly form of the recursive definition of the c_n's:

$$c_{n+2} = \frac{n(n+1)c_{n+1} - c_n}{(n+2)(n+1)} \;\; \textit{for all } n \geq 1, \;\; \textit{and} \;\; c_2 = -\frac{1}{2}c_0.$$

Clearly, $y(0) = 0$ implies that $c_0 = 0$ and $y'(0) = 1$ implies that $c_1 = 1$. We may now use the known value of c_0 and the second equation above to see that $c_2 = 0$. Then using the first equation with n = 1 and n = 2, plus a little arithmetical magic reveals successively that $c_3 = -1/6$ and $c_4 = -1/12$.

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9. (10 pts.) Transform the given initial value problem into an algebraic equation in $\mathfrak{L}\{y\}$ and solve for $\mathfrak{L}\{y\}$. Do not take inverse transforms and do not attempt to combine terms over a common denominator. Be very careful.

$$y''(x) + 3y'(x) + 2y = 0 \; ; \;\; y(0) = 1 \; , \;\; y'(0) = 2 \; .$$

Applying the Laplace transform operator to BOTH SIDES OF THE ODE, using the two initial conditions, and then solving for the transform of y should reveal that

$$\mathfrak{L}\{y\}(s) = \frac{s+5}{s^2+3s+2}\,.$$

[A common error: Failure to parenthesize the first derivative's transform correctly.]

Silly 10 Point Bonus: Compute the Laplace transform of the function

$$f(t) = \sin^3(bt).$$

Say where your work is, for it won't fit here.

I only have a hint: The needed trigonometric identity can be gotten with WD-40 or Eul. That's pronounced "oil" as in Euler.